SMALL SIMPLE QUASI-INJECTIVE MODULES

APICHART SA-NGUANNAM

A THESIS SUBMITTED IN PARTIAL FULLFILLMENT OF THE

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Name - Surname	Mr. Apichart Sa-nguannam
Program	Mathematics
Thesis Advisor	Assistant Professor Sarun Wongwai, Ph.D.
Academic Year	2012

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ABSTRACT

The purposes of this thesis are to (1) study properties and characterizations of small simple quasi-injective modules, (2) study properties and characterizations of endomorphism rings of small simple quasi-injective modules, (3) extend the concept of small principally quasi-injective modules, and (4) find some relations between small simple quasi-injective modules, small principally quasi-injective modules and projective modules.

Let R be a ring. A right R-module M is called *mininjective* if, for each simple right ideal K of R, every R-homomorphism $\gamma : K \to M$ extends to an R-homomorphism from R to M. A right R-module N is called *small principally M-injective* if every R-homomorphism from a small and principal submodule of M to N can be extended to an R-homomorphism from M to N. A right R-module M is called *small principally quasi-injective* if it is small principally M-injective. The notion of small principally quasi-injective modules is extended to be small simple quasi-injective modules. A right R-module N is called *small simple M-injective* if every R-homomorphism from a small and simple submodule of M to N can be extended to an R-homomorphism from M to N. A right R-module M is called *small simple quasi-injective* if it is small simple M-injective.

The results were as follows. (1) The following conditions are equivalent for a projective module M: (a) every small and simple submodule of M is projective; (b) every factor module of a small simple M-injective module is small simple M-injective; (c) every factor module of an injective R-module is small simple M-injective. (2) Let M be a right R-module and $S = End_R(M)$. Then the following conditions are equivalent: (a) M is small simple quasi-injective;

(b) if mR is small and simple, $m \in M$, then $l_M r_R(m) = Sm$; (c) if mR is small and simple and $r_R(m) \subset r_R(n)$, m, $n \in M$, then $Sn \subset Sm$; (d) if mR is small and simple, $m \in M$, then $l_M(r_R(m) \cap aR) = l_M(a) + Sm$ for all $a \in R$; (e) if mR is small and simple, $m \in M$, and $\gamma : mR \rightarrow M$ is an R-homomorphism, then $\gamma(m) \in Sm$. (3) Let M be a principal nonsingular module which is a principal self-generator and $Soc(M_R) \subset^e M$. If M is small simple quasi-injective, then $J(S) = 0$.

Keywords: Small Simple Quasi-injective Modules, Small Principally Quasi-injective Modules, Endomorphism Rings

หัวข้อวิทยานิพนธ์	มอดูลแบบสมอลซิมเปิลควอซี-อินเจคทีฟ
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บทคัดย่อ

วิทยานิพนธ์นี้มีวัตถุประสงค์เพื่อ (1) ศึกษาสมบัติและลักษณะเฉพาะของมอดูลแบบสมอลซิมเปิลควอซี-อินเจคทีฟ (2) ศึกษาสมบัติและลักษณะเฉพาะของริงอันตรสัณฐานของมอดูลแบบสมอลซิมเปิลควอซี-อินเจคทีฟ (3) ขยายแนวคิดของมอดูลแบบสมอลพรินซิแพ็ลลีควอซี-อินเจคทีฟ และ (4) หาความสัมพันธ์ระหว่างมอดูลแบบสมอลซิมเปิลควอซี-อินเจคทีฟ มอดูลแบบสมอลพรินซิแพ็ลลีควอซี-อินเจคทีฟและมอดูลแบบโปรเจคทีฟ

กำหนดให้ R เป็นริง จะเรียก R-มอดูลทางขวา M ว่า *มินอินเจคทีฟ* ก็ต่อเมื่อสำหรับแต่ละอุดมคติทางขวาแบบซิมเปิล K ของ R, ทุกๆ R-สาทิสสัณฐาน $\gamma : K \rightarrow M$ สามารถขยายไปยัง R-สาทิสสัณฐานจาก R ไปยัง M จะเรียก R-มอดูลทางขวา N ว่า *สมอลพรินซิแพ็ลลี M-อินเจคทีฟ* ก็ต่อเมื่อสำหรับแต่ละ R-สาทิสสัณฐานจากมอดูลย่อยแบบสมอลและพรินซิแพ็ลของ M ไปยัง N สามารถขยายไปยัง R-สาทิสสัณฐานจาก M ไปยัง N จะเรียก R-มอดูลทางขวา M ว่า *สมอลพรินซิแพ็ลลีควอซี-อินเจคทีฟ* ก็ต่อเมื่อ M เป็นสมอลพรินซิแพ็ลลี M-อินเจคทีฟ เราทำการขยายแนวคิดของมอดูลแบบสมอลพรินซิแพ็ลลีควอซี-อินเจคทีฟ มาเป็นมอดูลแบบสมอลซิมเปิลควอซี-อินเจคทีฟ โดยจะเรียก R-มอดูลทางขวา N ว่า *สมอลซิมเปิล M-อินเจคทีฟ* ก็ต่อเมื่อสำหรับแต่ละ R-สาทิสสัณฐานจากมอดูลย่อยแบบสมอลและซิมเปิลของ M ไปยัง N สามารถขยายไปยัง R-สาทิสสัณฐานจาก M ไปยัง N จะเรียก R-มอดูลทางขวา M ว่า *สมอลซิมเปิลควอซี-อินเจคทีฟ* ก็ต่อเมื่อ M เป็นสมอลซิมเปิล M-อินเจคทีฟ

ผลการวิจัยพบว่า (1) สำหรับมอดูลแบบโปรเจคทีฟ M จะได้ว่าเงื่อนไขดังต่อไปนี้มีความสมมูลกัน (a) ทุกๆมอดูลย่อยแบบสมอลและซิมเปิลของ M เป็นมอดูลแบบโปรเจคทีฟ (b) ทุกๆแฟคเตอร์มอดูลของมอดูลแบบสมอลซิมเปิล M-อินเจคทีฟ เป็นมอดูลแบบสมอลซิมเปิล M-อินเจคทีฟ (c) ทุกๆแฟคเตอร์มอดูลของ R-มอดูลแบบอินเจคทีฟ เป็นมอดูลแบบสมอลซิมเปิล M-อินเจคทีฟ

(2) กำหนดให้ M เป็น R-มอดูลทางขวาและ $S = End_R(M)$ เป็นริงอันตรสัณฐานของ M แล้วจะได้ว่าเงื่อนไขดังต่อไปนี้มีความสมมูลกัน (a) M เป็นสมอลซิมเปิลควอซี-อินเจคทีฟ (b) ถ้า mR เป็นสมอลและซิมเปิล โดยที่ $m \in M$, แล้วจะได้ว่า $l_M r_R(m) = Sm$. (c) ถ้า mR เป็นสมอลและซิมเปิล และ $r_R(m) \subset r_R(n)$ โดยที่ $m, n \in M$, แล้วจะได้ว่า $Sn \subset Sm$. (d) ถ้า mR เป็นสมอลและซิมเปิล โดยที่ $m \in M$, แล้วจะได้ว่า $l_M(r_R(m) \cap aR) = l_M(a) + Sm$ สำหรับทุกๆ $a \in R$. (e) ถ้า mR เป็นสมอลและซิมเปิล โดยที่ $m \in M$, และ $\gamma : mR \to M$ เป็น R-สาทิสสัณฐาน แล้วจะได้ว่า $\gamma(m) \in Sm$. (3) กำหนดให้ M เป็นมอดูลไม่เอกฐานแบบพรินซิเพิ็ล ซึ่งก่อกำเนิดตัวเองแบบพรินซิเพิ็ลและ $Soc(M_R) \subset^e M$ ถ้า M เป็นมอดูลแบบสมอลซิมเปิลควอซี-อินเจคทีฟ แล้วจะได้ว่า $J(S) = 0$.

คำสำคัญ: มอดูลแบบสมอลซิมเปิลควอซี-อินเจคทีฟ มอดูลแบบสมอลพรินซิเพิ็ลลีควอซี-อินเจคทีฟ ริงอันตรสัณฐาน

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Apichart Sa-nguannam

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List of Abbreviations

$A \oplus B$	A direct sum B
$End_R(M)$	The set of R-homomorphism from M to M called R-endomorphism of M
F	Field F
$f: M \to N$	A function f from M to N
$f(M)$	Image of f
$Hom_R(M,N)$	The set of R-homomorphism from M to N
$Im(f)$	Image of f
$J(M) = Rad(M_R)$	Jacobson radical of a right R-module M
$J(R) = Rad(R_R)$	Jacobson radical of a ring R
$J(S)$	Jacobson radical of a ring S
$J(S) \subset {}_S S_S$	$J(S)$ is an (two-side) ideal of ring S
$Ker(f)$	Kernel of f
$l_M(A)$	Left annihilator of A in M
M_R	M is a right R-module
$M_1 \times M_2$	Cartesian products of M_1 and M_2
M/K	A factor module of M modulo K or a factor module of M by K
$M \cong N$	M isomorphic N
R	Ring R
R_R	Ring R is a right R-module is called Regular right R-module
$Rej_M(\mathcal{U})$	Reject of $\mathcal{U}$ in M
R-module	Module over ring R
$r_R(X)$	Right annihilator of X in R
$Soc(M_R)$	Socle of module M
$Tr_M(\mathcal{U})$	Trace of $\mathcal{U}$ in M

List of Abbreviations (Continued)

$Z(M)$	Singular submodule of M
1_M	Identity map on set M
$\begin{pmatrix} F & F \\ F & F \end{pmatrix} = M_2(F)$	The set of all 2×2 matrices having elements of F as entries
$\eta : M \to M/K$	η (*eta*) is the natural epimorphism of M onto M/K
$\iota = \iota_{A \subset B} : A \to B$	ι (*iota*) is the inclusion map of A in B
$\varphi^{-1}(Ker(s))$	Inverse image of $Ker(s)$ under φ (*phi*)
π_j	π_j is the j-th projection map
$\forall$	For all
$\cap$	Intersection of set
$\not\subset$	is not subset
$\subset$	subset
$\in$	is in, member of set
$\subset^e$	Essential (Large) submodule
$\ll$	Superfluous (Small) submodule
$\prod_{i \in I} N_i$	Direct product of N_i
$\bigoplus_{i=1}^{n} N_i$	Direct sum of N_i

CHAPTER 1

INTRODUCTION

In modules and rings theory research field, there are three methods for doing the research. Firstly, to study about the fundamental of algebra and modules theory over arbitrary rings. Secondly, to study about the modules over special rings. Thirdly, to study about ring R by way of the categories of R-modules. Many mathematicians have concentrated on these methods.

1.1 Background and Statement of the Problems

Many generalizations of the injectivity were obtained, e.g. *principally injectivity* and *mininjectivity*. In [2], V. Camillo introduced the definition of principally injective modules by calling a right R-module M is *principally injective* if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M.

In [7], [8] and [9], Nicholson and Yousif studied to the structure of principally injective rings, mininjective modules and principally quasi-injective modules. They gave some applications of these rings and modules. From [7], a ring R is called *right principally injective* if every R-homomorphism from a principal right ideal of R to R can be extended to an R-homomorphism from R to R. From [8], a right R-module M is called *mininjective* if, for each simple right ideal K of R, every R-homomorphism $\gamma : K \to M$ extends to an R-homomorphism from R to M. Following from [9], they introduced the definition of principally quasi-injective modules by calling a right R-module M is *principally quasi-injective* if every R-homomorphism from a principal submodule of M to M can be extended to an R-endomorphism of M.

In [18] and [19], Sarun Wongwai introduced the definitions of small principally quasi-injective modules and quasi-small principally injective modules. Following from [18], a right R-module N is called *small principally M-injective* (briefly, *SP-M-injective*) if every R-homomorphism from a small and principal submodule of M to N can be extended to an R-homomorphism from M to N. A right R-module M is called *small principally quasi-injective* (briefly, *SPQ-injective*) if it is SP-M-injective.

Following from [19], a right R-module N is called *M-small principally-injective* (briefly, *M-small P-injective*) if every R-homomorphism from an M-cyclic small submodule of M to N can be extended to an R-homomorphism from M to N. A right R-module M is called *quasi-small principally-injective* (briefly, *quasi-small P-injective*) if it is M-small P-injective.

1.2 Purpose of the Study

In this thesis, we have the purposes of study which are to extend concept of the previous works and to generalize new concepts which are :

1.2.1 To extend the concept of *mininjective modules*.

1.2.2 To generalize the concept of *small principally quasi-injective modules*.

1.2.3 To establish and extend some new concepts which are dual to *small principally quasi-injective modules* [18] and q*uasi-small principally-injective modules* [19].

1.3 Research Questions and Hypothesis

We are interested in seeing to extend the characterizations and properties which remain valid from these previous concepts which can be extended from *principally injective modules* [2], *principally-injective rings* [7], *mininjective modules* [8], *principally quasi-injective modules* [9], *small principally quasi-injective modules* [18] and *quasi-small principally-injective modules* [19].

In this research, we introduce the definition of *small simple quasi-injective modules* and give characterizations and properties of these modules which are extended from the previous works.

By let M be a right R-module. A right R-module N is called *small simple M-injective* if every R-homomorphism from a small and simple submodule of M to N can be extended to an R-homomorphism from M to N. Dually, a right R-module M is called *small simple quasi-injective* if it is small simple M-injective. Many of results in this research are extended from *principally injective rings* [7], *mininjective rings* [8], *small principally quasi-injective modules* [18] and *quasi-small principally-injective modules* [19].

1.4 Theoretical Perspective

In this thesis, we use many of the fundamental theories which are concerned to the rings and modules research. By the concerned theories are :

1.4.1 The fundamental of algebra theories.

1.4.2 The basic properties of rings and modules theory.

1.5 Delimitations and Limitations of the Study

For this thesis, we have the scopes and the limitations of studying which are concerned to the previous works which are:

1.5.1 To extend the concept of *mininjective modules*.

1.5.2 To extend the concept of *small principally quasi-injective modules* and *quai-small P-injective modules*.

1.5.3 To characterize the concept in 1.5.2 and find some new properties.

1.6 Significance of the Study

The advantage of education and studying in this research, we can improve and develop the concepts and knowledge in the algebra and modules research field.

CHAPTER 2

LITERATURE REVIEW

In this chapter we give notations, definitions and fundamental theories of the modules and rings theory which are used in this thesis.

2.1 Rings, Modules, Submodules and Endomorphism Rings

This section is assembled summary of various notations, terminology and some background theories which are concerned and used for this thesis.

2.1.1 Definition. [14] By a *ring* we mean a nonempty set R with two binary operations + and $\cdot$, called *addition* and *multiplication* (also called *product*), respectively, such that

(1) $(R, +)$ is an additive abelian group.

(2) $(R, \cdot)$ is a multiplicative semigroup.

(3) Multiplication is distributive (on both sides) over addition; that is, for all $a, b, c \in R$, $a\cdot(b + c) = a\cdot b + a\cdot c$ and $(a + b)\cdot c = a\cdot c + b\cdot c$.

The two distributive laws are respectively called the *left distributive* law and the *right distributive* law.

A *commutative ring* is a ring R in which multiplication is commutative; i.e. if $a\cdot b = b\cdot a$ for all $a, b \in R$. If a ring is not commutative it is called *noncommutative*.

A *ring with unity* is a ring R in which the multiplicative semigroup $(R, \cdot)$ has an identity element; that is, there exists $e \in R$ such that $ea = a = ae$ for all $a \in R$. The element e is called *unity* or the *identity* element of R. Generally, the unity or identity element is denoted by 1.

In this thesis, R will be an associative ring with identity.

2.1.2 Definition. [14] A nonempty subset I of a ring R is called an *ideal* of R if

(1) $a, b \in I$ implies $a - b \in I$.

(2) $a \in I$ and $r \in R$ imply $ar \in I$ and $ra \in I$.

2.1.3 Definition. [13] A subgroup I of $(R, +)$ is called a *left ideal* of R if $RI \subset I$, and a *right ideal* if $IR \subset I$.

2.1.4 Definition. [14] A right ideal I of a ring R is called *principal* if $I = aR$ for some $a \in R$.

2.1.5 Definition. [14] Let R be a ring, M an additive abelian group and $(m, r) \mapsto mr$, a mapping of $M \times R$ into M such that

(1) $mr \in M$

(2) $(m_1 + m_2)r = m_1 r + m_2 r$

(3) $m(r_1 + r_2) = mr_1 + mr_2$

(4) $(mr_1)r_2 = m(r_1 r_2)$

(5) $m \cdot 1 = m$

for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. Then M is called a *right R-module*, often written as M_R.

Often mr is called the *scalar multiplication* or just *multiplication* of m by r on right. We define left R-module similarly.

2.1.6 Definition. [13] Let M be a right R-module. A subgroup N of $(M, +)$ is called a *submodule* of M if N is closed under multiplication with elements in R, that is $nr \in N$ for all $n \in N$, $r \in R$. Then N is also a right R-module by the operations induced from M :

$$N \times R \to N, (n, r) \mapsto nr, \text{ for all } n \in N, r \in R.$$

2.1.7 Proposition. *A subset N of an R-module M is a submodule of M if and only if*

(1) $0 \in N$.

(2) $n_1, n_2 \in N$ *implies* $n_1 - n_2 \in N$.

(3) $n \in N, r \in R$ *implies* $nr \in N$.

Proof. See [15, Lemma 5.3]. □

2.1.8 Definition. [1] Let M be a right R-module and let K be a submodule of M. Then the set of cosets

$$M/K = \{ x + K \mid x \in M \}$$

is a right R-module relative to the addition and scalar multiplication defined via

$$(x+K) + (y+K) = (x+y) + K \quad \text{and} \quad (x+K)r = xr + K.$$

The additive identity and inverses are given by

$$K = 0 + K \quad \text{and} \quad -(x+K) = -x + K.$$

The module M/K is called (the *right R-factor module of*) *M modulo K* or the *factor module of M by K*.

2.1.9 Definition. [13] Let M and N be right R-modules. A function $f : M \longrightarrow N$ is called an (R-*module*) *homomorphism* if for all $m, m_1, m_2 \in M$ and $r \in R$

$$f(m_1 r + m_2) = f(m_1)r + f(m_2).$$

Equivalently, $f(m_1 + m_2) = f(m_1) + f(m_2)$ and $f(mr) = f(m)r$.

The set of R-homomorphisms of M in N is denoted by $Hom_R(M,N)$. In particular, with this addition and the composition of mappings, $Hom_R(M,M) = End_R(M)$ becomes a ring, called the *endomorphism ring* of M.

2.1.10 Definition. [1] Let $f : M \longrightarrow N$ be an R-homomorphism. Then

(1) f is called R-*monomorphism* (or R-monic) if f is injective (one-to-one).

(2) f is called R-*epimorphism* (or R-epic) if f is surjective (onto).

(3) f is called R-*isomorphism* if f is bijective (one-to-one and onto).

Two modules M and N are said to be R-*isomorphic*, abbreviated $M \cong N$ in case there is an R-*isomorphism* $f : M \longrightarrow N$.

Note: An R-homomorphism $f : M \longrightarrow M$ is called an R-*endomorphism*.

2.1.11 Definition. [1] Let K be a submodule of M. Then the mapping $\eta_K : M \longrightarrow M/K$ from M onto the factor module M/K defined by

$$\eta_K(x) = x + K \in M/K \qquad (x \in M)$$

is seen to be an R-epimorphism with kernel K. We call η_K the *natural epimorphism of M onto M/K.*

2.1.12 Definition. [1] Let $A \subset B$. Then the function $\iota = \iota_{A \subset B} : A \longrightarrow B$ defined by $\iota = (1_{B|A}) : a \mapsto a$ for all $a \in A$ is called the *inclusion map* of A in B. Note that if $A \subseteq B$ and $A \subseteq C$, and if $B \neq C$, then $\iota_{A \subseteq B} \neq \iota_{A \subseteq C}$. Of course $1_A = \iota_{A \subseteq A}$.

2.1.13 Definition. [14] Let M and N be right R-modules and let $f : M \longrightarrow N$ be an R-homomorphism. Then the set

$Ker(f) = \{\, x \in M \mid f(x) = 0 \,\}$ is called the *kernel* of f

and

$f(M) = \{\, f(x) \in N \mid x \in M \,\}$ is called the *homomorphic image* (or simply *image*) of M under f and is denoted by $Im(f)$.

2.1.14 Proposition. *Let M and N be right R-modules and let $f : M \longrightarrow N$ be an R-homomorphism. Then*

(1) *$Ker(f)$ is a submodule of M.*

(2) *$Im(f) = f(M)$ is a submodule of N.*

Proof. See [13, 6.5]. □

2.1.15 Proposition. *Let M and N be right R-modules and let $f : M \longrightarrow N$ be an R-isomorphism. Then the inverse mapping $f^{-1} : N \longrightarrow M$ is an R-isomorphism.*

Proof. See [14, Chapter 14, 3]. □

2.2 Essential and Superfluous Submodules

In this section, we give the definitions of essential and superfluous submodules and some theories which are used in this thesis.

2.2.1 Definition. [13] A submodule K of M is called *essential* (or *large*) in M, abbreviated $K \subset^{e} M$, if for every submodule L of M, $K \cap L = 0$ implies $L = 0$.

2.2.2 Definition. [13] A submodule K of M is called *superfluous* (or *small*) in M, abbreviated $K \ll M$, if for every submodule L of M, $K + L = M$ implies $L = M$.

2.2.3 Proposition. *Let M be a right R-module with submodules $K \subset N \subset M$ and $H \subset M$. Then*

(1) $N \ll M$ *if and only if* $K \ll M$ *and* $N/K \ll M/K$;

(2) $H + K \ll M$ *if and only if* $H \ll M$ *and* $K \ll M$.

Proof. See [1, Proposition 5.17]. □

2.2.4 Proposition. *If $K \ll M$ and $f : M \to N$ is a homomorphism then $f(K) \ll N$. In particular, if $K \ll M \subset N$ then $K \ll N$.*

Proof. See [1, Proposition 5.18]. □

2.3 Annihilators and Singular Modules

In this section, we give the definitions of annihilators, singular modules and some theories which are used in this thesis.

2.3.1 Definition. [1] Let M be a right (resp. left) R-module. For each $X \subset M$, the *right* (resp. *left*) *annihilator* of X in R is defined by

$$r_R(X) = \{ r \in R \mid xr = 0, \forall x \in X \} \text{ (resp. } l_R(X) = \{ r \in R \mid rx = 0, \forall x \in X \}).$$

For a singleton $\{x\}$, we usually abbreviated to $r_R(x)$ (resp. $l_R(x)$).

2.3.2 Proposition. *Let M be a right R-module, let X and Y be subsets of M and let A and B be subsets of R. Then*

(1) $r_R(X)$ *is a right ideal of* R.

(2) $X \subset Y$ *imples* $r_R(Y) \subset r_R(X)$.

(3) $A \subset B$ *imples* $l_M(B) \subset l_M(A)$.

(4) $X \subset l_M r_R(X)$ *and* $A \subset r_R l_M(A)$.

Proof. See [1, Proposition 2.14 and Proposition 2.15]. □

2.3.3 Proposition. *Let* M *and* N *be right* R*-module*s *and let* $f : M \rightarrow N$ *be a homomorphism. If* N' *is an essential submodule of* N, *then* $f^{-1}(N')$ *is an essential submodule of* M.

Proof. See [4, Lemma 5.8(a)]. □

2.3.4 Proposition. *Let* M *be a right* R*-module over an arbitrary ring* R, *the set*

$$Z(M) = \{\, x \in M \mid r_R(x) \text{ is essential in } R_R \,\}$$

is a submodule of M.

Proof. See [4, Lemma 5.9]. □

2.3.5 Definition. [4] The submodule $Z(M) = \{\, x \in M \mid r_R(x)$ is essential in $R_R \,\}$ is called the *singular submodule* of M. The module M is called a *singular module* if $Z(M) = M$. The module M is called *a nonsingular module* if $Z(M) = 0$.

2.4 Maximal and Minimal Submodules

In this section, we give the definitions and some properties of maximal submodules, minimal (or simple) submodules and some theories which are used in this thesis.

2.4.1 Definition. [13] A right R-module M is called *simple* if $M \neq 0$ and M has no submodules except 0 and M.

2.4.2 Definition. [13] A submodule K of M is called *maximal submodule* of M if $K \neq M$ and it is not properly contained in any proper submodules of M, i.e. K is *maximal in* M if, $K \neq M$ and for every $A \subset M$, $K \subset A$ implies $K = A$.

2.4.3 Definition. [13] A submodule N of M is called *minimal* (or *simple*) *submodule* of M if $N \neq 0$ and it has no non-zero proper submodules of M, i.e. N is *minimal* (or *simple*) *in M* if $N \neq 0$ and for every non-zero submodule A of M, $A \subset N$ implies $A = N$.

2.4.4 Proposition. *Let M and N be right R-modules. If* $f : M \rightarrow N$ *is an epimorphism with* $Ker(f) = K$, *then there is a unique isomorphism* $\sigma : M/K \rightarrow N$ *such that* $\sigma(m+K) = f(m)$ *for all* $m \in M$.

Proof. See [1, Corollary 3.7]. □

2.4.5 Proposition. *Let K be a submodule of M. A factor module M/K is simple if and only if K is a maximal submodule of M.*

Proof. See [1, Corollary 2.10]. □

2.5 Injective and Projective Modules

In this section, we give the definitions of the injective modules, injective testing, projective modules and some theories which are used in this thesis.

2.5.1 Definition. [1] Let M be a right R-module. A right R-module U is called *injective relative to M* (or *U is M-injective*) if for every submodule K of M, for every homomorphism $\varphi : K \rightarrow U$ can be extended to a homomorphism $\alpha : M \rightarrow U$.

A right R-module U is said to be *injective* if it is M-injective for every right R-module M.

2.5.2 Proposition. *The following statements about a right R-module U are equivalent* :

(1) *U is injective*;

(2) *U is injective relative to R*;

(3) *For every right ideal* $I \subset R_R$ *and every homomorphism* $h : I \rightarrow U$ *there exists an* $x \in U$ *such that h is left multiplicative by x*

$$h(a) = xa \text{ for all } a \in I.$$

Proof. See [1, 18.3, Baer's Criterion]. □

2.5.3 Definition. [1] Let M be a right R-module. A right R-module U is called *projective relative to M* (or *U is M-projective*) if for every N_R, every epimorphism $g : M_R \to N_R$, for every homomorphism $\gamma: U_R \to N_R$ can be lifted to an R-homomorphism $\hat{\gamma} : U \to M$.

A right R-module U is said to be *projective* if it is projective for every right R-module M.

2.5.4 Proposition. *Every right* (*resp. left*) *R-module can be embedded in an injective right* (*resp. left*) *R-module.*

Proof. See [1, Proposition 18.6]. □

2.6 Direct Summands and Product of Modules

Given two modules M_1 and M_2 we can construct their Cartesian product $M_1 \times M_2$. The structure of this product module is then determined "co-ordinatewise" from the factors $M_1 \times M_2$. For this section we give the definitions of direct summand, the projection and the injection maps, product of modules and some theories which are used in this thesis.

2.6.1 Definition. [1] Let M be a right R-module. A submodule X of M is called a *direct summand* of M if there is a submodule Y of M such that $X \cap Y = 0$ and $X + Y = M$. We write $M = X \oplus Y$; such that Y is also a *direct summand*.

2.6.2 Definition. [1] Let M_1 and M_2 be R-modules. Then with their products module $M_1 \times M_2$ are associated the natural injections and projections

$$\varphi_j : M_j \to M_1 \times M_2 \quad \text{and} \quad \pi_j : M_1 \times M_2 \to M_j$$

($j = 1, 2$), are defined by

$$\varphi_1(x_1) = (x_1, 0), \qquad \varphi_2(x_2) = (0, x_2)$$

and

$$\pi_1(x_1, x_2) = x_1, \qquad \pi_2(x_1, x_2) = x_2.$$

Moreover, we have

$$\pi_1\varphi_1 = 1_{M_1} \quad \text{and} \quad \pi_2\varphi_2 = 1_{M_2}.$$

2.6.3 Definition. [1] Let A be a direct summand of M with complementary direct summand B, so $M = A \oplus B$. Then

$$\pi_A : a + b \mapsto a \qquad (a \in A, b \in B)$$

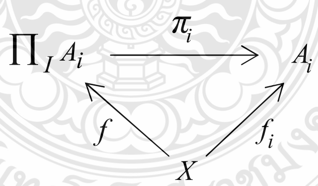
defines an epimorphism $\pi_A : M \to A$ is called *the projection of M on A along B*.

2.6.4 Definition. [13] Let $\{A_i, i \in I\}$ be a family of objects in the category C. An object P in C with morphisms $\{\pi_i : P \to A_i\}$ is called the *product* of the family $\{A_i, i \in I\}$ if :

For every family of morphisms $\{f_i : X \to A_i\}$ in the category C, there is a unique morphism $f : X \to P$ with $\pi_i f = f_i$ for all $i \in I$.

For the object P, we usually write $\prod_{i \in I} A_i$, $\prod_I A_i$ or $\prod A_i$. If all A_i are equal to A, then we put $\prod_I A_i = A^I$.

The morphism π_i are called the *i-projections* of the product. The definition can be described by the following commutative diagram :



2.6.5 Definition. [13] Let $\{M_i, i \in I\}$ be a family of R-modules and $(\prod_{i \in I} M_i, \pi_i)$ the product of the M_i . For $m, n \in \prod_{i \in I} M_i$, $r \in R$, using

$$\pi_i(m + n) = \pi_i(m) + \pi_i(n) \quad \text{and} \quad \pi_i(mr) = \pi_i(m)r,$$

a right R-module structure is defined on $\prod_{i \in I} M_i$ such that the π_i are homomorphisms. With this structure $(\prod_{i \in I} M_i, \pi_i)$ is the product of the $\{M_i, i \in I\}$ in R-module.

2.6.6 Proposition. *Properties*:

(1) *If* $\{f_i : N \to M_i, i \in I\}$ *is a family of morphisms, then we get the map*

$$f : N \to \prod_{i \in I} M_i \quad \text{such that} \quad n \mapsto (f_i(n))_{i \in I},$$

and $Ker(f) = \bigcap_I Ker(f_i)$ *since* $f(n) = 0$ *if and only if* $f_i(n) = 0$ *for all* $i \in I$.

(2) *For every* $j \in I$, *we have a canonical embedding*

$$\varepsilon_j : M_j \to \prod_{i \in I} M_i, \quad \text{such that} \quad m_j \mapsto (m_j \delta_{ji})_{i \in I}, m_j \in M_j,$$

with $\varepsilon_j \pi_j = 1_{M_j}$, *i.e.* π_j *is a retraction and* ε_j *a coretraction.*

This construction can be extended to larger subsets of I : *For a subset* $A \subset I$ *we form the product* $\prod_{i \in A} M_i$ *and a family of homomorphisms*

$$f_j : \prod_{i \in A} M_i \to M_j, \quad f_j = \begin{cases} \pi_j \text{ for } j \in A, \\ 0 \text{ for } j \in I - A. \end{cases}$$

Then there is a unique homomorphism

$$\varepsilon_A : \prod_{i \in A} M_i \to \prod_{i \in I} M_i \text{ with } \varepsilon_A \pi_j = \begin{cases} \pi_j \text{ for } j \in A, \\ 0 \text{ for } j \in I - A. \end{cases}$$

The universal property of $\prod_{i \in A} M_i$ *yields a homomorphism*

$$\pi_A : \prod_{i \in I} M_i \to \prod_{i \in A} M_i \text{ with } \pi_A \pi_j = \pi_j \text{ for } j \in A.$$

Together this implies $\varepsilon_A \pi_A \pi_j = \varepsilon_A \pi_j = \pi_j$ *for all* $j \in I$, *and by the properties of the product* $\prod_{i \in A} M_i$, *we get* $\varepsilon_A \pi_A = 1_{M_A}$.

Proof. See [13, 9.3, Properties (1), (2)] □

2.7 Generated and Cogenerated Classes

In this section, we give some definitions and theories of the generated and cogenerated classes which are concerned in this thesis.

2.7.1 Definition. [13] A subset X of a right R-module M is called a *generating set* of M if $XR = M$. We also say that X *generates* M or M is *generated by* X. If there is a finite generating set in M, then M is called *finitely generated*.

2.7.2 Definition. [1] Let $\mathcal{U}$ be a class of right R-modules. A module M is (*finitely*) *generated by* $\mathcal{U}$ (or $\mathcal{U}$ (*finitely*) *generates* M) if there exists an epimorphism

$$\bigoplus_{i \in I} U_i \to M$$

for some (finite) set I and $U_i \in \mathcal{U}$ for every $i \in I$.

If $\mathcal{U} = \{U\}$ is a singleton, then we say that M is (*finitely*) *generated by* $\mathcal{U}$ or (*finitely*) *U-generates*; this means that there exists an epimorphism

$$U^{(I)} \to M$$

for some (finite) set I.

2.7.3 Proposition. *If a module M has a generating set $L \subset M$, then there exists an epimorphism*

$$R^{(L)} \to M$$

Moreover, M is finitely R-generated if and only if M is finitely generated.

Proof. See [1, Theorem 8.1]. □

2.7.4 Definition. [17] Let M be a right R-module. A submodule N of M is said to be an *M-cyclic submodule* of M if it is the image of an endomorphism of M.

2.7.5 Definition. [1] Let $\mathcal{U}$ be a class of right R-modules. A module M is (*finitely*) *cogenerated by* $\mathcal{U}$ (or $\mathcal{U}$ (*finitely*) *cogenerates* M) if there exists a monomorphism

$$M \to \prod_{i \in I} U_i$$

for some (finite) set I and $U_i \in \mathcal{U}$ for every $i \in I$.

If $\mathcal{U} = \{ U \}$ is a singleton, then we say that a module M is (*finitely*) *cogenerated by* $\mathcal{U}$ or (*finitely*) *U-cogenerates*; this means that there exists a monomorphism

$$M \rightarrow U^I$$

for some (finite) set I.

2.8 The Trace and Reject

In this section, we give some definitions and theories of the trace and reject which are concerned in this thesis.

2.8.1 Definition. [1] Let $\mathcal{U}$ be a class of right R-modules. The *trace* of $\mathcal{U}$ in M and the *reject* of $\mathcal{U}$ in M are defined by

$$Tr_M(\mathcal{U}) = \sum \{ \, Im(h) \mid h : U \rightarrow M \text{ for some } U \in \mathcal{U} \, \}$$

and

$$Rej_M(\mathcal{U}) = \bigcap \{ \, Ker(h) \mid h : M \rightarrow U \text{ for some } U \in \mathcal{U} \, \}.$$

If $\mathcal{U} = \{ U \}$ is a singleton, then the trace of $\mathcal{U}$ in M and the reject of $\mathcal{U}$ in M are in the form

$$Tr_M(U) = \sum \{ \, Im(h) \mid h \in Hom_R(U, M) \, \}$$

and

$$Rej_M(U) = \bigcap \{ \, Ker(h) \mid h \in Hom_R(M, U) \, \}.$$

2.8.2 Proposition. *Let $\mathcal{U}$ be a class of right R-modules and let M be a right R-module. Then*

(1) *$Tr_M(\mathcal{U})$ is the unique largest submodule L of M generated by $\mathcal{U}$;*

(2) *$Rej_M(\mathcal{U})$ is the unique smallest submodule K of M such that M/K is cogenerated by $\mathcal{U}$.*

Proof. See [1, Proposition 8.12]. □

2.9 Socle and Radical of Modules

In this section, we give some definitions and theories of the socle and radical of modules which are used in this thesis.

2.9.1 Definition. [13] Let M be a right R-module. The *socle* of M, $Soc(M)$, we denote the sum of all simple submodules of M. If there are no simple submodules in M we put $Soc(M) = 0$.

2.9.2 Definition. [13] Let M be a right R-module. The *radical* of M, $Rad(M)$, we denote the intersection of all maximal submodules of M. If M has no maximal submodules we set $Rad(M) = M$.

2.9.3 Proposition. *Let $\mathcal{E}$ be the class of simple R-modules and let M be an R-module. Then*

$$\begin{aligned} Soc(M) &= Tr_M(\mathcal{E}) \\ &= \bigcap \{ L \subset M \mid L \text{ is essential in } M \}. \end{aligned}$$

Proof. See [13, 21.1]. □

2.9.4 Proposition. *Let $\mathcal{E}$ be the class of simple R-modules and let M be an R-module. Then*

$$\begin{aligned} Rad(M) &= Rej_M(\mathcal{E}) \\ &= \sum \{ L \subset M \mid L \text{ is superfluous in } M \}. \end{aligned}$$

Proof. See [13, 21.5]. □

2.9.5 Proposition. *Let M be a right R-module. A right R-module M is finitely generated if and only if $Rad(M) \ll M$ and $M/Rad(M)$ is finitely generated.*

Proof. See [13, 21.6, (4)]. □

2.9.6 Proposition. *Let M be a right R-module. Then $Soc(M) \subset^e M$ if and only if every non-zero submodule of M contains a minimal submodule.*

Proof. See [1, Corollary 9.10]. □

2.10 The Radical of a Ring

In this section, we give some definitions and theories of the radical of a ring which are used in this thesis.

2.10.1 Definition. [1] Let R be a ring. The radical $Rad(R_R)$ of R_R is an (two side) ideal of R. This ideal of R is called the (*Jacobson*) *radical* of R, and we usually abbreviated by

$$J(R) = Rad(R_R).$$

2.10.2 Definition. [1] Let R be a ring. An element $x \in R$ is called *right* (*left*) *quasi-regular* if $1 - x$ has a right (resp. left) inverse in R.

An element $x \in R$ is called *quasi-regular* if it is right and left quasi-regular.

A subset of R is said to be (*right*, *left*) *quasi-regular* if every element in it has the corresponding property.

2.10.3 Proposition. *Given a ring R for each of the following subsets of R is equal to the radical $J(R)$ of R.*

(J_1) *The intersection of all maximal right* (*left*) *ideals of R;*

(J_2) *The intersection of all right* (*left*) *primitive ideals of R;*

(J_3) $\{ x \in R \mid rxs$ *is quasi-regular for all* $r, s \in R \}$;

(J_4) $\{ x \in R \mid rx$ *is quasi-regular for all* $r \in R \}$;

(J_5) $\{ x \in R \mid xs$ *is quasi-regular for all* $s \in R \}$;

(J_6) *The union of all the quasi-regular right* (*left*) *ideals of R;*

(J_7) *The union of all the quasi-regular ideals of R;*

(J_8) *The unique largest superfluous right* (*left*) *ideals of R;*

Moreover, (J_3), (J_4), (J_5), (J_6) *and* (J_7) *also describe the radical $J(R)$ if "quasi-regular" is replaced by "right quasi-regular" or by "left quasi-regular".*

Proof. See [1, Theorem 15.3]. □

2.10.4 Proposition. *Let R be a ring with radical $J(R)$. Then for every right R-module M,*

$$J(R)M_R \subset Rad(M_R).$$

If R is semisimple modulo its radical, then for every right R-module,

$$J(R)M_R = Rad(M_R)$$

and $M/J(R)M_R$ is semisimple.

Proof. See [1, Corollary 15.18]. □

CHAPTER 3

RESEARCH RESULT

In this chapter, we present the results of small simple M-injective modules and small simple quasi-injective modules.

3.1 Small Simple M-injective Modules

3.1.1 Definition. Let M be a right R-module. A right R-module N is called *small simple M-injective* if every R-homomorphism from a small and simple submodule of M to N can be extended to an R-homomorphism from M to N.

3.1.2 Lemma. *Let M and N be right R-modules. Then N is small simple M-injective if and only if for each small and simple submodule mR of M,*

$$l_N r_R(m) = Hom_R(M,N)m.$$

Proof. $(\Rightarrow)$ Let N be a small simple M-injective module and let mR be a small and simple submodule of M. To show that $l_N r_R(m) = Hom_R(M,N)m$. $(\supset)$ Let $\varphi(m) \in Hom_R(M,N)m$. To show that $\varphi(m) \in l_N r_R(m)$, i.e. $\varphi(m)r = 0$, for every $r \in r_R(m)$. Let $r \in r_R(m)$. Then $mr = 0$. Hence $\varphi(m)r = \varphi(mr) = \varphi(0) = 0$. $(\subset)$ Let $x \in l_N r_R(m)$. To show that $x \in Hom_R(M,N)m$. Define $\varphi : mR \to xR$ by $\varphi(mr) = xr$ for every $r \in R$. Let $mr_1, mr_2 \in mR$ such that $mr_1 = mr_2$. Then $mr_1 - mr_2 = 0$, hence $m(r_1 - r_2) = 0$, so $r_1 - r_2 \in r_R(m)$. Since $x \in l_N r_R(m)$, $x(r_1 - r_2) = 0$. It follows that $xr_1 = xr_2$. Thus $\varphi(mr_1) = xr_1 = xr_2 = \varphi(mr_2)$. This shows that φ is well-defined. Let $mr_1, mr_2 \in mR$ and $r \in R$. Then $\varphi(mr_1r + mr_2) = \varphi(m(r_1r + r_2)) = x(r_1r + r_2) = xr_1r + xr_2 = (xr_1)r + xr_2 = \varphi(mr_1)r + \varphi(mr_2)$. This shows that φ is an R-homomorphism. Since N is small simple M-injective, there exists an R-homomorphism

$\hat{\varphi}: M \to N$ such that $\hat{\varphi}\iota_1 = \iota_2\varphi$ where ι_1: $mR \to M$ and ι_2: $xR \to N$ are the inclusion maps. Then $x = x\cdot 1 = \varphi(m\cdot 1) = \varphi(m) = \iota_2\varphi(m) = \hat{\varphi}\iota_1(m) = \hat{\varphi}(m) \in Hom_R(M,N)m$.

($\Leftarrow$) To show that N is small simple M-injective. Let mR be a small and simple submodule of M and let $\varphi: mR \to N$ be an R-homomorphism. Let $r \in r_R(m)$. Then $mr = 0$. Hence $\varphi(m)r = \varphi(mr) = \varphi(0) = 0$. This shows that $\varphi(m) \in l_N r_R(m)$. By assumption, we have $\varphi(m) \in Hom_R(M,N)m$. Then $\varphi(m) = \hat{\varphi}(m)$ for some $\hat{\varphi} \in Hom_R(M,N)$. Hence $\varphi(m) = \hat{\varphi}(m) = \hat{\varphi}\iota(m)$ where $\iota: mR \to M$ is the inclusion map. This shows that $\hat{\varphi}$ is an extension of φ. □

3.1.3 Example. *Let* $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ *where* F *is a field*, $M_R = R_R$ *and* $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. *Then N is small simple M-injective*.

Proof. We have only $X_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, $X_3 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $X_4 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$, $X_5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $X_6 = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ are R-submodules of M. We have non-zero submodule of M two sets are $X_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. We found $X_1 \ll M$ because for every $X_n \subset M$, $2 \le n \le 5$, $X_n \ne M$ then $X_1 + X_n \ne M$. We found X_2 is not small in M because $X_2 + X_3 = M$. Let $m = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in X_1$. Then $mR = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = X_1$. Hence $mR = X_1$. This shows that X_1 is a simple submodule of M. Let $\varphi: X_1 \to N$ be an R-homomorphism. Since $1 \in F$, we have $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in X_1$, there exists x_{11}, $x_{12} \in F$ such that $\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}$. Then $\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}$, so $x_{11} = 0$. Define $\hat{\varphi}: M \to N$ by $\hat{\varphi}\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} x_{12}a & x_{12}b \\ 0 & 0 \end{pmatrix}$ for every $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Let $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ such that $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$. Then $\hat{\varphi}\left(\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\right) = \begin{pmatrix} x_{12}a_1 & x_{12}b_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{12}a_2 & x_{12}b_2 \\ 0 & 0 \end{pmatrix} = \hat{\varphi}\left(\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\right)$. This shows that $\hat{\varphi}$ is well-defined.

Let $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then $\hat{\varphi}\left(\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\right) =$ $\hat{\varphi}\left(\begin{pmatrix} a_1x & a_1y + b_1z \\ 0 & c_1z \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix} a_1x + a_2 & a_1y + b_1z + b_2 \\ 0 & c_1z + c_2 \end{pmatrix}\right) = \begin{pmatrix} x_{12}(a_1x + a_2) & x_{12}(a_1y + b_1z + b_2) \\ 0 & 0 \end{pmatrix} =$ $\begin{pmatrix} x_{12}a_1x + x_{12}a_2 & x_{12}a_1y + x_{12}b_1z + x_{12}b_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{12}a_1x & x_{12}a_1y + x_{12}b_1z \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} x_{12}a_2 & x_{12}b_2 \\ 0 & 0 \end{pmatrix} =$ $\begin{pmatrix} x_{12}a_1 & x_{12}b_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} + \begin{pmatrix} x_{12}a_2 & x_{12}b_2 \\ 0 & 0 \end{pmatrix} = \hat{\varphi}\left(\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\right)\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} + \hat{\varphi}\left(\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\right)$.
This shows that $\hat{\varphi}$ is an R-homomorphism. To show that $\varphi = \hat{\varphi}\iota$. Let $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in X_1$.
Then $\varphi\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & x_{12}x \\ 0 & 0 \end{pmatrix}$.
Hence $\hat{\varphi}\iota\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & x_{12}x \\ 0 & 0 \end{pmatrix}$. This shows that $\hat{\varphi}$ is an extension of φ. □

3.1.4 Proposition. *Let M be a right R-module and let* $\{ N_i , i \in I \}$ *be a family of right R-modules. Then the direct product* $\prod_{i \in I} N_i$ *is small simple M-injective if and only if each* N_i *is small simple M-injective.*

Proof. ($\Rightarrow$) Let $\pi_i : \prod_{i \in I} N_i \to N_i$ and $\varphi_i : N_i \to \prod_{i \in I} N_i$, for each $i \in I$, be the i-th projection and the i-th injection maps, respectively. To show that each $i \in I$, N_i is small simple M-injective. Let $i \in I$, mR a small simple submodule of M and let $\varphi : mR \to N_i$ be an R-homomorphism. Then by assumption, there exists an R-homomorphism $\hat{\varphi} : M \to \prod_{i \in I} N_i$ such that $\varphi_i\varphi = \hat{\varphi}\,\iota$ where $\iota : mR \to M$ is the inclusion map. Hence $\pi_i\varphi_i\varphi = \pi_i\hat{\varphi}\,\iota$, so by Definition 2.6.2, $\varphi = \pi_i\hat{\varphi}\,\iota$. Thus $\pi_i\hat{\varphi}$ is an extension of φ.

($\Leftarrow$) Let mR be a small and simple submodule of M and let $\varphi : mR \to \prod_{i \in I} N_i$ be an R-homomorphism. Since for each $i \in I$, N_i is small simple M-injective, there exits an R-homomorphism $\alpha_i : M \to N_i$ such that $\pi_i\varphi = \alpha_i\iota$ where $\iota : mR \to M$ is the inclusion map and $\pi_i : \prod_{i \in I} N_i \to N_i$ is the i-th projection map. Then by Definition 2.6.5 and Proposition 2.6.6, we obtain $\hat{\varphi}: M \to \prod_{i\in I} N_i$ such that $\pi_i\hat{\varphi} = \alpha_i$. Hence $\pi_i\hat{\varphi}\,\iota = \alpha_i\iota$, so $\pi_i\varphi = \alpha_i\iota = \pi_i\hat{\varphi}\,\iota$. Thus $\pi_i\varphi = \pi_i\hat{\varphi}\,\iota$. Therefore $\varphi = \hat{\varphi}\,\iota$. □

3.1.5 Lemma. *Let N_i $(1 \le i \le n)$ be small simple M-injective modules. Then $\bigoplus_{i=1}^{n} N_i$ is small simple M-injective.*

Proof. Assume that for each $1 \le i \le n$, N_i is small simple M-injective. To show that $\bigoplus_{i=1}^{n} N_i$ is small simple M-injective. Let mR be a small and simple submodule of M and let $\varphi : mR \to \bigoplus_{i=1}^{n} N_i$ be an R-homomorphism. Since for each $1 \le i \le n$, N_i is small simple M-injective, there exists an R-homomorphism $\varphi_i : M \to N_i$ such that $\pi_i\varphi = \varphi_i\,\iota$ where $\iota : mR \to M$ is the inclusion map and $\pi_i : \bigoplus_{i=1}^{n} N_i \to N_i$ is the i-projection map. Set $\hat{\varphi} = \iota_1\varphi_1 + \iota_2\varphi_2 + \ldots + \iota_n\varphi_n : M \to \bigoplus_{i=1}^{n} N_i$ where $\iota_i : N_i \to \bigoplus_{i=1}^{n} N_i$ for each $1 \le i \le n$ is the i-injection map. To show that $\varphi = \hat{\varphi}\,\iota$. Let $mr \in mR$. Then $\hat{\varphi}\,\iota(mr) = \hat{\varphi}(mr) = \iota_1\varphi_1(mr) + \iota_2\varphi_2(mr) + \ldots + \iota_n\varphi_n(mr) = \varphi_1(mr) + \varphi_2(mr) + \ldots + \varphi_n(mr) = \pi_1\varphi(mr) + \pi_2\varphi(mr) + \ldots + \pi_n\varphi(mr) = (\pi_1 + \pi_2 + \ldots + \pi_n)\varphi(mr) = \varphi(mr)$. Hence $\hat{\varphi}$ is an extension of φ. □

3.1.6 Lemma. *Any direct summand of a small simple M-injective module is again small simple M-injective module.*

Proof. Let N be a small simple M-injective module and let A be a direct summand of N. To show that A is small simple M-injective. Let mR be a small and simple submodule of M and let $\varphi : mR \to A$ be an R-homomorphism. Let $\varphi_A : A \to N$ be the injection map. Since N is small simple M-injective, there exists an R-homomorphism $\hat{\varphi} : M \to N$ such that $\varphi_A\varphi = \hat{\varphi}\,\iota$ where $\iota : mR \to M$ is the inclusion map. Let $\pi_A : N \to A$ be the projection map. Then $\pi_A\varphi_A\varphi = \pi_A\hat{\varphi}\,\iota$. Hence by Definition 2.6.2, $\varphi = \pi_A\hat{\varphi}\,\iota$. This shows that $\pi_A\hat{\varphi}$ is an extension of φ. □

3.1.7 Theorem. *The following conditions are equivalent for a projective module M.*

(1) *Every small and simple submodule of M is projective.*

(2) *Every factor module of a small simple M-injective module is small simple M-injective.*

(3) *Every factor module of an injective R-module is small simple M-injective.*

Proof. (1) $\Rightarrow$ (2) Let N be a small simple M-injective module, X a submodule of N, mR a small and simple submodule of M and let $\varphi: mR \rightarrow N/X$ be an R-homomorphism. Since mR is projective, there exists an R-homomorphism $\alpha : mR \rightarrow N$ such that $\varphi = \eta\alpha$ where $\eta : N \rightarrow N/X$ is the natural R-epimorphism. Since N is small simple M-injective, there exists an R-homomorphism $\beta : M \rightarrow N$ such that $\alpha = \beta\iota$ where $\iota : mR \rightarrow M$ is the inclusion map. Then $\varphi = \eta\alpha = \eta\beta\iota$. Hence $\varphi = \eta\beta\iota$. This shows that $\eta\beta$ is an extension of φ. Thus N/X is small simple M-injective.

(2) $\Rightarrow$ (3) Let N be an injective R-module and X be a submodule of N. Then by (2), N/X is small simple M-injective.

(3) $\Rightarrow$ (1) Let mR be a small and simple submodule of M, $\alpha : A \rightarrow B$ an R-epimorphism and let $\varphi: mR \rightarrow B$ be an R-homomorphism. Let E be an injective R-module and embed A in E by Proposition 2.5.4. Since α is an R-epimorphism, by Proposition 2.4.4, there exists an R-isomorphism $\sigma: A/Ker(\alpha) \rightarrow B$ such that $\alpha = \sigma\eta_1$ where $\eta_1: A \rightarrow A/Ker(\alpha)$ is the natural R-epimorphism. Then by Proposition 2.1.15, we have $\sigma^{-1}: B \rightarrow A/Ker(\alpha)$ is an R-isomorphism, so $B \cong A/Ker(\alpha)$ and $A/Ker(\alpha) \subset E/Ker(\alpha)$. By assumption, there exists an R-homomorphism $\hat{\varphi}: M \rightarrow E/Ker(\alpha)$ such that $\iota_1\sigma^{-1}\varphi = \hat{\varphi}\iota_2$ where $\iota_1: A/Ker(\alpha) \rightarrow E/Ker(\alpha)$ and $\iota_2: mR \rightarrow M$ are the inclusion maps. Since M is projective, there exists an R-homomorphism $\beta : M \rightarrow E$ such that $\hat{\varphi} = \eta_2\beta$ where $\eta_2: E \rightarrow E/Ker(\alpha)$ is the natural R-epimorphism. Then $\hat{\varphi}\iota_2 = \eta_2\beta\iota_2$. Hence $\iota_1\sigma^{-1}\varphi = \hat{\varphi}\iota_2 = \eta_2\beta\iota_2$. It follows that $\iota_1\sigma^{-1}\varphi = \eta_2\beta\iota_2$. To show that $\beta(mR) \subset A$. Let $mx \in mR$. Then $\iota_1\sigma^{-1}\varphi(mx) = \eta_2\beta\iota_2(mx) = \eta_2\beta(mx) = \eta_2(\beta(mx)) = \beta(mx) + Ker(\alpha)$. Hence $\iota_1\sigma^{-1}\varphi(mx) = \sigma^{-1}\varphi(mx) = a + Ker(\alpha)$ for some $a \in A$,

so $\beta(mx) + Ker(\alpha) = a + Ker(\alpha)$. Thus $\beta(mx) - a \in Ker(\alpha)$. It follows that $\beta(mx) = (\beta(mx) - a) + a \in Ker(\alpha) + A = A$. To show that $\varphi = \alpha\beta$. Let $mx \in mR$. Then $\iota_1\sigma^{-1}\varphi(mx) = \sigma^{-1}\varphi(mx) = \eta_2\beta\iota_2(mx) = \eta_2\beta(mx)$. Hence $\iota_1\sigma^{-1}\varphi(mx) = \eta_2\beta(mx) = \beta(mx) + Ker(\alpha)$, so $\iota_1\sigma^{-1}\varphi(mx) = \beta(mx) + Ker(\alpha)$. Since α is an R-epimorphism, $\varphi(mx) = \alpha(a)$ for some $a \in A$. Thus $\iota_1\sigma^{-1}\varphi(mx) = \iota_1\sigma^{-1}\alpha(a) = \sigma^{-1}\alpha(a) = \eta_1(a) = a + Ker(\alpha)$. It follows that $\beta(mx) + Ker(\alpha) = a + Ker(\alpha)$. Then $\beta(mx) - a \in Ker(\alpha)$. Hence $\alpha(\beta(mx) - a) = 0$, so $\alpha\beta(mx) = \alpha(a) = \varphi(mx)$. Thus $\alpha\beta(mx) = \varphi(mx)$. This shows that β lifts φ. □

3.2 Small Simple Quasi-injective Modules

A right R-modules M is called *small simple quasi-injective* if it is small simple M-injective. Write $S = End_R(M)$ denoted the endomorphism ring of M. In this section, we present the results of characterizations and properties of small simple quasi-injective modules.

3.2.1 Lemma. *Let M be a right R-module and $S = End_R(M)$. Then the following conditions are equivalent :*

(1) *M is small simple quasi-injective.*

(2) *If mR is small and simple, $m \in M$, then $l_M r_R(m) = Sm$.*

(3) *If mR is small and simple and $r_R(m) \subset r_R(n)$, $m, n \in M$, then $Sn \subset Sm$.*

(4) *If mR is small and simple, $m \in M$, then $l_M(r_R(m) \cap aR) = l_M(a) + Sm$ for all $a \in R$.*

(5) *If mR is small and simple, $m \in M$, and $\gamma : mR \longrightarrow M$ is an R-homomorphism, then $\gamma(m) \in Sm$.*

Proof. (1) $\Rightarrow$ (2) Let mR be small and simple and let $m \in M$. To show that $l_M r_R(m) = Sm$. ($\supset$) Let $\varphi(m) \in Sm$. To show that $\varphi(m) \in l_M r_R(m)$, i.e. $\varphi(m)r = 0$, for every $r \in r_R(m)$.

Let $r \in r_R(m)$. Then $mr = 0$. Hence $\varphi(m)r = \varphi(mr) = \varphi(0) = 0$. ($\subset$) Let $x \in l_M r_R(m)$. To show that $x \in Sm$. Define $\varphi : mR \to xR$ by $\varphi(mr) = xr$ for every $r \in R$. Let $mr_1, mr_2 \in mR$ such that $mr_1 = mr_2$. Then $mr_1 - mr_2 = 0$, hence $m(r_1 - r_2) = 0$, so $r_1 - r_2 \in r_R(m)$. Since $x \in l_M r_R(m)$, $x(r_1 - r_2) = 0$. It follows that $xr_1 = xr_2$. Thus $\varphi(mr_1) = xr_1 = xr_2 = \varphi(mr_2)$. This shows that φ is well-defined. Let $mr_1, mr_2 \in mR$ and $r \in R$. Then $\varphi(mr_1r + mr_2) = \varphi(m(r_1r + r_2)) = x(r_1r + r_2) = xr_1r + xr_2 = (xr_1)r + xr_2 = \varphi(mr_1)r + \varphi(mr_2)$. This shows that φ is an R-homomorphism. Since M is small simple quasi-injective, there exists an R-homomorphism $\hat{\varphi} : M \to M$ such that $\iota_1\varphi = \hat{\varphi}\iota_2$ where $\iota_1 : xR \to M$ and $\iota_2 : mR \to M$ are the inclusion maps. Then $x = x\cdot 1 = \varphi(m\cdot 1) = \varphi(m) = \iota_1\varphi(m) = \hat{\varphi}\iota_2(m) = \hat{\varphi}(m) \in Sm$.

(2) $\Rightarrow$ (1) To show that M is small simple quasi-injective. Let mR be a small and simple submodule of M and let $\varphi : mR \to M$ be an R-homomorphism. Let $r \in r_R(m)$. Then $mr = 0$. Hence $\varphi(m)r = \varphi(mr) = \varphi(0) = 0$. This shows that $\varphi(m) \in l_M r_R(m)$. Then by assumption, we have $\varphi(m) \in Sm$. Hence $\varphi(m) = \hat{\varphi}(m)$ for some $\hat{\varphi} \in S$. Thus $\varphi(m) = \hat{\varphi}(m) = \hat{\varphi}\iota(m)$. This shows that $\varphi = \hat{\varphi}\iota$.

(2) $\Rightarrow$ (3) Let mR be small and simple and let $r_R(m) \subset r_R(n)$, $m, n \in M$. To show that $Sn \subset Sm$. Let $x \in l_M r_R(n)$. To show that $x \in l_M r_R(m)$. Let $a \in r_R(m)$. Since $r_R(m) \subset r_R(n)$, $a \in r_R(n)$, so $xa = 0$. Thus $x \in l_M r_R(m)$. This shows that $l_M r_R(n) \subset l_M r_R(m)$. Let $\varphi(n) \in Sn$. To show that $\varphi(n) \in l_M r_R(n)$, i.e. $\varphi(n)r = 0$, for every $r \in r_R(n)$. Let $r \in r_R(n)$. Then $nr = 0$. Hence $\varphi(n)r = \varphi(nr) = \varphi(0) = 0$. This shows that $Sn \subset l_M r_R(n)$. It follows that $Sn \subset l_M r_R(n) \subset l_M r_R(m) = Sm$.

(3) $\Rightarrow$ (4) Let mR be small and simple, $m \in M$ and let $a \in R$. To show that $l_M\big(r_R(m) \cap aR\big) = l_M(a) + Sm$. ($\subset$) Let $x \in l_M\big(r_R(m) \cap aR\big)$. To show that $x \in l_M(a) + Sm$. Since $x \in l_M\big(r_R(m) \cap aR\big)$, $x\big(r_R(m) \cap aR\big) = 0$. Hence $xar = 0$ every $r \in R$

such that $mar = 0$, so $r \in r_R(ma)$. Let $b \in r_R(ma)$. Then $mab = 0$. Hence $xab = 0$, so $b \in r_R(xa)$. This shows that $r_R(ma) \subset r_R(xa)$. Since $mar = 0$, we show two cases, i.e. $ma = 0$ and $ma \neq 0$. If $ma = 0$, then $mar = 0$ every $r \in R$. Hence $r \in r_R(ma)$, so $r \in r_R(xa)$. Thus $xar = 0$ every $r \in R$. Since we have $1 \in R$, $xa = xa\cdot 1 = 0$. Therefore $xa = 0$. It follows that $x \in l_M(a) \subset l_M(a) + Sm$. If $ma \neq 0$, then $maR \neq 0$. We have $aR \subset R_R$, so $maR \subset mR$. Since mR is simple, $maR = mR$. This shows that maR is a small and simple submodule of M. By (3), we have $Sxa \subset Sma$. Then $xa = 1_M(xa) \in Sxa \subset Sma$. Hence $xa \in Sma$, so $xa = \varphi(ma)$ for some $\varphi \in S$. Thus $xa - \varphi(ma) = 0$. Therefore $(x - \varphi(m))a = 0$. It follows that $x - \varphi(m) \in l_M(a)$. Then $x = (x - \varphi(m)) + \varphi(m) \in l_M(a) + Sm$. $(\supset)$ Let $x \in l_M(a) + Sm$. To show that $x \in l_M\big(r_R(m) \cap aR\big)$, i.e. $xay = 0$, for every $y \in R$ such that $may = 0$. Since $x \in l_M(a) + Sm$, $x = v + \varphi(m)$ for some $v \in l_M(a)$, $\varphi \in S$. Then $xa = va + \varphi(m)a = 0 + \varphi(m)a$, hence $xa = \varphi(m)a$. Let $y \in R$ such that $may = 0$. Thus $xay = \varphi(m)ay = \varphi(may) = \varphi(0) = 0$.

(4) $\Rightarrow$ (2) Let mR be a small and simple submodule of M. We have $1_R \in R$. Put $a = 1_R$, then by (4), $l_M r_R(m) = Sm$.

(3) $\Rightarrow$ (5) Let mR be a small and simple submodule of M and let $\gamma : mR \longrightarrow M$ be an R-homomorphism. To show that $\gamma(m) \in Sm$. Let $x \in r_R(m)$. Then $mx = 0$. Hence $\gamma(m)x = \gamma(mx) = \gamma(0) = 0$, so $x \in r_R(\gamma(m))$. This shows that $r_R(m) \subset r_R(\gamma(m))$. Then by (3), we have $S\gamma(m) \subset Sm$. It follows that $\gamma(m) = 1_M\,\gamma(m) \in S\gamma(m) \subset Sm$.

(5) $\Rightarrow$ (1) To show that M is small simple quasi-injective. Let mR be a small and simple submodule of M and let $\varphi : mR \longrightarrow M$ be an R-homomorphism. Then by (5), $\varphi(m) \in Sm$. Hence $\varphi(m) = \hat{\varphi}(m)$ for some $\hat{\varphi} \in S$. This shows that $\hat{\varphi}$ is an extension of φ. □

3.2.2 Lemma. *Let M be a small simple quasi-injective module and $S = End_R(M)$. If $m \in M$ and $\alpha \in S$ with $\alpha(M)$ is small and simple, then*

$$l_S(Ker(\alpha) \cap mR) = l_S(m) + S\alpha.$$

Proof. ($\supset$) Let $x \in l_S(m) + S\alpha$. To show that $x \in l_S(Ker(\alpha) \cap mR)$, i.e. $xmy = 0$, for every $y \in R$ such that $\alpha(my) = 0$. Since $x \in l_S(m) + S\alpha$, $x = v + \varphi\alpha$ for some $v \in l_S(m)$, $\varphi \in S$. Then $xm = v(m) + \varphi\alpha(m) = 0 + \varphi\alpha(m)$. Hence $xm = \varphi\alpha(m)$. Let $y \in R$ such that $\alpha(my) = 0$. Thus $xmy = \varphi\alpha(m)y = \varphi\alpha(my) = \varphi(\alpha(my)) = \varphi(0) = 0$.

($\subset$) Let $\beta \in l_S(Ker(\alpha) \cap mR)$. To show that $\beta \in l_S(m) + S\alpha$. Let $b \in r_R(\alpha(m))$. Then $\alpha(m)b = \alpha(mb) = 0$. Hence $mb \in Ker(\alpha) \cap mR$, so $\beta(mb) = 0$. Thus $b \in r_R(\beta(m))$. This shows that $r_R(\alpha(m)) \subset r_R(\beta(m))$. Then by Proposition 2.3.2, $l_M r_R(\beta(m)) \subset l_M r_R(\alpha(m))$. If $\alpha(m) = 0$, then $\alpha(m)r = 0$ every $r \in R$. Then $r \in r_R(\alpha(m))$. Hence $r \in r_R(\beta(m))$, so $\beta(m)r = 0$ every $r \in R$. We have $1 \in R$, so $\beta(m) = \beta(m)\cdot 1 = 0$. Thus $\beta(m) = 0$. Therefore $\beta \in l_S(m)$. It follows that $\beta \in l_S(m) \subset l_S(m) + S\alpha$. If $\alpha(m) \neq 0$, then $\alpha(m)R \neq 0$. Since α is an R-homomorphism, $\alpha(m)R = \alpha(mR) \subset \alpha(M)$. Since $\alpha(M)$ is simple in M and $\alpha(m)R \neq 0$, $\alpha(m)R = \alpha(M)$. This shows that $\alpha(m)R$ is a small and simple submodule of M. Then by Lemma 3.2.1, we have $l_M r_R(\alpha(m)) = S\alpha(m)$. Let $f\beta(m) \in S\beta(m)$. To show that $f\beta(m) \in l_M r_R(\beta(m))$, i.e. $f\beta(m)r = 0$, for every $r \in r_R(\beta(m))$. Let $r \in r_R(\beta(m))$. Then $\beta(m)r = \beta(mr) = 0$. Hence $f\beta(m)r = f\beta(mr) = f(\beta(mr)) = f(0) = 0$. This shows that $S\beta(m) \subset l_M r_R(\beta(m))$. Then $S\beta(m) \subset l_M r_R(\beta(m)) \subset l_M r_R(\alpha(m)) = S\alpha(m)$. Hence $S\beta(m) \subset S\alpha(m)$, so $\beta(m) = 1_M\beta(m) \in S\beta(m) \subset S\alpha(m)$. Thus $\beta(m) \in S\alpha(m)$. Therefore $\beta(m) = \gamma\alpha(m)$ for some $\gamma \in S$. It follows that $\beta(m) - \gamma\alpha(m) = 0$. Then $(\beta - \gamma\alpha)(m) = 0$. Hence $\beta - \gamma\alpha \in l_S(m)$. Thus $\beta = (\beta - \gamma\alpha) + \gamma\alpha \in l_S(m) + S\alpha$. □

Following [9], a right R-module M is called a *principal self-generator* if every element $m \in M$ has the form $m = \gamma(m_1)$ for some $\gamma : M \longrightarrow mR$.

3.2.3 Proposition. *Let M be a principal module which is a principal self-generator and let $S = End_R(M)$. Then the following conditions are equivalent* :

(1) *M is small simple quasi-injective.*

(2) *$l_S(Ker(\alpha) \cap mR) = l_S(m) + S\alpha$ for all $m \in M$ and $\alpha \in S$ with $\alpha(M)$ is small and simple in M.*

(3) *$l_S(Ker(\alpha)) = S\alpha$ for all $\alpha \in S$ with $\alpha(M)$ is small and simple in M.*

(4) *$Ker(\alpha) \subset Ker(\beta)$,* where *$\alpha, \beta \in S$ with $\alpha(M)$ is small and simple in M, implies $S\beta \subset S\alpha$.*

Proof. (1) $\Rightarrow$ (2) By lemma 3.2.2.

(2) $\Rightarrow$ (3) Write $M = m_0R$ for some $m_0 \in M$. Put $m = m_0$ in (2). Then $l_S(Ker(\alpha) \cap m_0R) = l_S(m_0) + S\alpha$. We have $Ker(\alpha) \cap m_0R = Ker(\alpha)$ and $l_S(m_0) = 0$, so $l_S(Ker(\alpha)) = S\alpha$.

(3) $\Rightarrow$ (4) Let $\alpha, \beta \in S$ with $\alpha(M)$ is small and simple in M and $Ker(\alpha) \subset Ker(\beta)$. To show that $S\beta \subset S\alpha$. Since $Ker(\alpha) \subset Ker(\beta)$, by Proposition 2.3.2, $l_S Ker(\beta) \subset l_S Ker(\alpha)$. Let $\varphi\beta \in S\beta$. To show that $\varphi\beta \in l_S Ker(\beta)$, i.e. $\varphi\beta(x) = 0$, for every $x \in Ker(\beta)$. Let $x \in Ker(\beta)$. Then $\beta(x) = 0$, hence $\varphi\beta(x) = \varphi(\beta(x)) = \varphi(0) = 0$. This shows that $S\beta \subset l_S Ker(\beta)$. Thus by (3), $S\beta \subset l_S Ker(\beta) \subset l_S Ker(\alpha) = S\alpha$.

(4) $\Rightarrow$ (1) Let mR be a small and simple submodule of M and let $\varphi : mR \longrightarrow M$ be an R-homomorphism. Since M is a principal self-generator module, by [9] there exists $\beta \in S$ such that $\beta(m_1) = m$ for some $\beta : M \longrightarrow mR$. Then $\beta(m_1R) = \beta(m_1)R = mR$. Since $\beta(M) \subset mR$ and we have $m_1R \subset M$, $\beta(m_1R) \subset \beta(M)$. Then $mR = \beta(m_1R) \subset \beta(M)$. It follows that $\beta(M) = mR$. This shows that $\beta(M)$ is a small and simple submodule of M. Let $x \in Ker(\beta)$.

Then $\varphi\beta(x) = \varphi(\beta(x)) = \varphi(0) = 0$. Hence $x \in Ker(\varphi\beta)$. This shows that $Ker(\beta) \subset Ker(\varphi\beta)$. Thus by (4), $S\varphi\beta \subset S\beta$. We have $1_M \in S$, so $\varphi\beta = 1_M\varphi\beta \in S\varphi\beta \subset S\beta$. It follows that $\varphi\beta \in S\beta$. Then $\varphi\beta = \hat{\varphi}\beta$ for some $\hat{\varphi} \in S$. To show that $\varphi = \hat{\varphi}\iota$. Let $mx \in mR$. Then $\varphi(mx) = \varphi(m)x = \varphi(\beta(m_1))x = \varphi(\beta(m_1)x) = \varphi\beta(m_1x) = \hat{\varphi}\beta(m_1x) = \hat{\varphi}\beta(m_1)x = \hat{\varphi}(m)x = \hat{\varphi}(mx) = \hat{\varphi}\iota(mx)$. □

3.2.4 Theorem. *Let M be a small simple quasi-injective module, $m, n \in M$ and mR is small and simple,*

(1) *If mR embeds in nR, then Sm is an image of Sn.*

(2) *If nR is an image of mR, then Sn embeds in Sm.*

(3) *If $mR \cong nR$, then $Sm \cong Sn$.*

Proof. (1) Let $f : mR \to nR$ be an R-monomorphism. Since M is small simple quasi-injective, there exists an R-homomorphism $\hat{f} : M \to M$ such that $\iota_2 f = \hat{f}\iota_1$ where $\iota_1 : mR \to M$ *and* $\iota_2 : nR \to M$ are the inclusion maps. Define $\sigma : Sn \to Sm$ by $\sigma(\alpha(n)) = \alpha\hat{f}(m)$ for every $\alpha \in S$. Let $0 = \alpha(n) \in Sn$. Since $f(mR) \subset nR$, $\alpha f(mR) \subset \alpha(nR)$, so $\alpha f(m) = \alpha f(m\cdot 1) \in \alpha f(mR) \subset \alpha(nR) = \alpha(n)R = 0\cdot R = 0$. Then $\sigma(\alpha(n)) = \alpha\hat{f}(m) = \alpha f(m) = 0$. This shows that σ is well-defined. Let $\alpha_1(n), \alpha_2(n) \in Sn$ and $s \in S$. Then $\sigma(s\alpha_1(n) + \alpha_2(n)) = \sigma((s\alpha_1 + \alpha_2)n) = (s\alpha_1 + \alpha_2)\hat{f}(m) = s\alpha_1\hat{f}(m) + \alpha_2\hat{f}(m) = s\sigma(\alpha_1(n)) + \sigma(\alpha_2(n))$. This shows that σ is an S-homomorphism. If $f = 0$, then $f(mx) = 0$ for every $mx \in mR$. Hence f is not an R-monomorphism, a contradiction. Thus $f \neq 0$. We have $0 \neq \hat{f}(mR) = f(mR) \subset M$. Let $mx \in mR$. Then $\hat{f}(mx) \in \hat{f}(mR)$. Hence $\hat{f}(mx) = f(mx) \in f(mR) = f(m)R$, so $\hat{f}(mx) \in f(m)R$. This shows that $\hat{f}(mR) \subset f(m)R$. Thus $\hat{f}(mR) = f(mR) = f(m)R$. It follows that $\hat{f}(mR) = f(m)R$. Thus by Definition 2.4.3, $f(m)R$ is simple in M. By Proposition 2.2.4, $f(m)R = \hat{f}(m)R \ll M$. Therefore $f(m)R$ is small and simple in M. Let $x \in r_R(f(m))$. Then $f(mx) = f(m)x = 0$. Hence $mx \in Ker(f)$. Since f is an R-monomorphism, $mx = 0$, so $x \in r_R(m)$. This shows that

$r_R(f(m)) \subset r_R(m)$. By lemma 3.2.1, $Sm \subset Sf(m)$. We have $1_M \in S$, so $m = 1_M(m) \in Sm \subset Sf(m)$ so $m \in Sf(m)$. Then $m = \alpha f(m)$ for some $\alpha \in S$. To show that σ is an S-epimorphism. Since $m = \alpha f(m) \in Sf(m)$ and $\alpha f(m) = \alpha \hat{f}(m) = \sigma(\alpha(n)) \in \sigma(Sn)$, $m \in Sf(m) \subset \sigma(Sn)$, so $Sm \subset \sigma(Sn)$. It follows that $Sm = \sigma(Sn)$.

(2) Let $f: mR \rightarrow nR$ be an R-epimorphism. We have $n\cdot 1 \in nR$, so $n = n\cdot 1 = f(my)$ for some $y \in R$. Since M is small simple quasi-injective, there exists an R-homomorphism $\hat{f}: M \rightarrow M$ such that $\iota_2 f = \hat{f}\iota_1$ where $\iota_1: mR \rightarrow M$ *and* $\iota_2: nR \rightarrow M$ are the inclusion maps. Define $\sigma : Sn \rightarrow Sm$ by $\sigma(\alpha(n)) = \alpha\hat{f}(my)$ for every $\alpha \in S$. Let $0 = \alpha(n) \in Sn$. Then $\sigma(\alpha(n)) = \alpha\hat{f}(my) = \alpha f(my) = \alpha(n) = 0$. This shows that σ is well-defined. Let $\alpha_1(n)$, $\alpha_2(n) \in Sn$ and $s \in S$. Then $\sigma(s\alpha_1(n) + \alpha_2(n)) = \sigma((s\alpha_1 + \alpha_2)n) = (s\alpha_1 + \alpha_2)\hat{f}(my) = s\alpha_1\hat{f}(my) + \alpha_2\hat{f}(my) = s\sigma(\alpha_1(n)) + \sigma(\alpha_2(n))$. This shows that σ is an S-homomorphism. To show that σ is an S-monomorphism, i.e. $Ker(\sigma) = \{0\}$. ($\supset$) It is clear. ($\subset$) Let $\alpha(n) \in Ker(\sigma)$. Then $\sigma(\alpha(n)) = 0$. Hence $0 = \sigma(\alpha(n)) = \alpha\hat{f}(my) = \alpha f(my) = \alpha(n)$. It follows that $\alpha(n) = 0 \in \{0\}$.

(3) It is clear by (1) and (2). □

3.2.5 Proposition. *Let M be a principal module which is a principal self-generator. If M is small simple quasi-injective, then* $Soc(M_R) \subset r_M(J(S))$.

Proof. Let mR be a simple submodule of M. To show that $mR \subset r_M(J(S))$, i.e. $\alpha(m) = 0$, for every $\alpha \in J(S)$. Let $\alpha \in J(S)$. Suppose $\alpha(m) \neq 0$. Since M is principal self-generator, $mR = \sum_{s\in I} s(M)$ for some $I \subset S$ by [17, Proposition 2.7]. Since mR is simple, there exists $0 \neq s \in I \subset S$ such that $s(M) = mR$. Thus $\alpha s \neq 0$. To show that $Ker(s) = Ker(\alpha s)$. Let $x \in Ker(s)$. Then $s(x) = 0$. Hence $\alpha s(x) = \alpha(s(x)) = \alpha(0) = 0$. Thus $x \in Ker(\alpha s)$. This shows that $Ker(s) \subset Ker(\alpha s)$. By Proposition 2.4.4, we have $M/Ker(s) \cong s(M)$, so $M/Ker(s)$ is simple in M. Hence by Proposition 2.4.5, $Ker(s)$ is maximal in M. Thus $Ker(s) = Ker(\alpha s)$.

Define $f : s(M) \to \alpha s(M)$ by $f(s(m)) = \alpha s(m)$ for every $m \in M$. Let $0 = s(m) \in s(M)$. Then $f(s(m)) = \alpha s(m) = \alpha(s(m)) = \alpha(0) = 0$. This shows that f is well-defined. Let $s(m_1)$, $s(m_2) \in s(M)$ and $r \in R$. Then $f(s(m_1)r + s(m_2)) = f(s(m_1 r) + s(m_2)) = f(s(m_1 r + m_2)) = \alpha s(m_1 r + m_2) = \alpha s(m_1 r) + \alpha s(m_2) = \alpha s(m_1)r + \alpha s(m_2) = f(s(m_1))r + f(s(m_2))$. This shows that f is an R-homomorphism. Let $\alpha(s(m)) \in \alpha s(M)$. We see that f is an R-epimorphism because every $\alpha(s(m)) \in \alpha s(M)$, we have $s(m) \in s(M)$ such that $f(s(m)) = \alpha s(m)$. If $0 \neq Ker(f)$, then $0 \neq Ker(f) \subset s(M)$. Since $s(M)$ is simple, $Ker(f) = s(M)$, a contradiction. Hence $0 = Ker(f)$, so f is an R-monomorphism. Thus f is an R-isomorphism, $s(M) \cong \alpha s(M)$. Therefore $\alpha s(M)$ is simple in M. Since M is a principal module, by Proposition 2.9.5, $J(M) \ll M$. By Proposition 2.10.4, $J(S)M \subset J(M)$. Hence $J(S)M \subset J(M) \ll M$, so by Proposition 2.2.3, $J(S)M \ll M$. Since $\alpha \in J(S)$ and $J(S) \subset {}_S S_S$, $J(S)S \subset J(S)$, so $\alpha s \in J(S)S \subset J(S)$. Then $\alpha s \in J(S)$. Hence $\alpha s(M) \subset J(S)M \ll M$, so $\alpha s(M) \ll M$. Thus $\alpha s(M)$ is a small and simple submodule of M. Since M is small simple quasi-injective, by Proposition 3.2.3, we have $l_S(Ker(\alpha s)) = S\alpha s$, so $l_S(Ker(s)) = S\alpha s$. We have $s \in l_S(Ker(s))$, so $s \in S\alpha s$. It follows that $s = \beta\alpha s$ for some $\beta \in S$. Then $s - \beta\alpha s = 0$. Hence $(1 - \beta\alpha)s = 0$, so by Proposition 2.10.3, $(1 - \beta\alpha)$ has a right inverse. Thus $(1 - \beta\alpha)^{-1} \cdot (1 - \beta\alpha)s = (1 - \beta\alpha)^{-1} \cdot 0$. Therefore $s = (1 - \beta\alpha)^{-1} \cdot 0 = 0$, a contradiction. □

Let M be a right R-module with $S = End_R(M)$. Following [6], we write a symbol delta is denoted by $\Delta = \{ s \in S \mid Ker(s) \subset^e M \}$. It is known that Δ is an ideal of S [6, Lemma 3.2].

3.2.6 Proposition. *Let M be a principal module which is a principal self-generator and $Soc(M_R) \subset^e M$. If M is small simple quasi-injective, then $J(S) \subset \Delta$.*

Proof. Let $s \in J(S)$. To show that $s \in \Delta$, i.e. $Ker(s) \subset^e M$. If $Ker(s) \not\subset^e M$, then there exists a non-zero submodule N of M such that $Ker(s) \cap N = 0$. Since $Soc(M_R) \subset^e M$, by Proposition 2.9.6, there exists a simple submodule mR of M such that $mR \subset Soc(M_R) \cap N$. Since M is principal self-generator and mR is simple, $mR = t(M)$ for some $0 \neq t \in S$ by [17, Proposition 2.9]. By the similar proof of Proposition 3.2.5, we have $Ker(t) = Ker(st)$, so $t(M) \cong st(M)$ and $st(M) \ll M$. Thus $st(M)$ is a small and simple submodule of M. Since M is small simple quasi-injective, by Proposition 3.2.3, we have $l_S(Ker(st)) = Sst$, so $l_S(Ker(t)) = Sst$. We have $t \in l_S(Ker(t))$, so $t \in Sst$. Therefore $t = \alpha st$ for some $\alpha \in S$. Then $t - \alpha st = 0$. Hence $(1 - \alpha s)t = 0$, so by Proposition 2.10.3, $(1 - \alpha s)$ has a right inverse. Thus $(1 - \alpha s)^{-1}\cdot(1 - \alpha s)t = (1 - \alpha s)^{-1}\cdot 0$. It follows that $t = (1 - \alpha s)^{-1}\cdot 0 = 0$, a contradiction. □

3.2.7 Proposition. *Let M be a principal nonsingular module which is a principal self-generator and $Soc(M_R) \subset^e M$. If M is small simple quasi-injective, then $J(S) = 0$.*

Proof. By Proposition 3.2.6, we have $J(S) \subset \Delta$, we show that $\Delta = 0$. Let $s \in \Delta$. To show that $s = 0$. Let $m \in M$. Define $\varphi : R \to M$ by $\varphi(r) = mr$ for every $r \in R$. Let $0 = r \in R$. Then $\varphi(r) = mr = m\cdot 0 = 0$. This shows that φ is well-defined. Let $r_1, r_2 \in R$ and $r \in R$. Then $\varphi(r_1 r + r_2) = m(r_1 r + r_2) = mr_1 r + mr_2 = (mr_1)r + mr_2 = \varphi(r_1)r + \varphi(r_2)$. This shows that φ is an R-homomorphism. We have

$$\begin{aligned} r_R(s(m)) &= \{ r \in R \mid s(m)r = 0 \} \\ &= \{ r \in R \mid s(mr) = 0 \} \\ &= \{ r \in R \mid mr \in Ker(s) \} \\ &= \{ r \in R \mid \varphi(r) \in Ker(s) \} \\ &= \varphi^{-1}(Ker(s)). \end{aligned}$$

Since $s \in \Delta$, $Ker(s) \subset^e M$. Then by Proposition 2.3.3, we have $\varphi^{-1}(Ker(s)) \subset^e R$. Hence $r_R(s(m)) = \varphi^{-1}(Ker(s)) \subset^e R$, so $r_R(s(m)) \subset^e R$. Thus by Definition 2.3.5, $s(m)$ is an element of the singular submodule $Z(M)$ of M. Since M is a nonsingular module, by Definition 2.3.5, $Z(M) = 0$, so $s(m) = 0$. As this is true for all $m \in M$, we have $s = 0$. Therefore $\Delta = 0$ as required. □

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Appendix

Conference Proceeding

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26–27 เมษายน พ.ศ. 2555

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88 ถ.รองเมือง เขตปทุมวัน กรุงเทพฯ

จัดโดย

ศูนย์ส่งเสริมการวิจัยคณิตศาสตร์แห่งประเทศไทย
สมาคมคณิตศาสตร์แห่งประเทศไทย ในพระบรมราชูปถัมภ์
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยมหิดล

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Small Simple Quasi-injective Modules

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Abstract : Let M be a right R–module. A right R-module N is called *small simple M-injective* if, every R-homomorphism from a small and simple submodule of M to N can be extended to an R-homomorphism from M to N. In this paper, we give some characterizations and properties of small simple quasi-injective modules.

Keywords : small principally quasi-injective modules; small simple quasi-injective modules; endomorphism rings.

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1. Introduction

Let R be a ring. A right R-module M is called *mininjective* [8] if, for each simple right ideal K of R, every R-homomorphism $\gamma : K \to M$ extends to R; equivalently, if $\gamma = m\cdot$ is left multiplication by some element m of M. Following [9], a right R-module M is called *principally quasi-injective module* if every R-homomorphism from a principal submodule of M to M can be extended to an endomorphism of M. In [15], S. Wongwai, introduced the definition of small principally quasi-injective modules, a right R- module N is called *small principally M-injective (or SP-M-injective)* if, every R-homomorphism from a small and principal submodule of M to N can be extended to an R-homomorphism from M to N. A right R-module M is called small principally quasi-injective (briefly, SPQ-injective) if it is SP-M-injective. In this note we introduce the definition of small simple quasi-injective modules and give some characterizations and properties. Some results on principally quasi-injective modules [9] are extended to these modules.

Throughout this paper, R will be an associative ring with identity and all modules are unitary right R-modules. For right R-modules M and N, $\mathrm{Hom}_R(M, N)$ denotes the set of all R-homomorphisms from M to N and $S = End_R(M)$ denotes the endomorphism ring of M. If X is a subset of M the right (resp. left) annihilator of X in R (resp. S) is denoted by $r_R(X)$ (resp. $l_S(X)$). By notations, $N \subset^{\oplus} M$, $N \subset^{e} M$, and $N \ll M$ we mean that N is a direct summand, an essential submodule and a superfluous submodule of M, respectively. We denote the Jacobson radical of M by $J(M)$.

2. Small Simple Quasi-injective Modules

Following [1], a submodule K of a right R-module M is *superfluous* (or *small*) in M, abbreviated $K \ll M$, in case for every submodule L of M, $K + L = M$ implies $L = M$. It is clear that $kR \ll R$ if and only if $k \in J(R)$. A right R-module M is *simple* in case $M \neq 0$ and it has no non-trivial submodules.

Definition 2.1. Let M be a right R-module. A right R-module N is called *small simple M-injective* if, every R-homomorphism from a small and simple submodule of M to N can be extended to an R-homomorphism from M to N.

Lemma 2.2. *Let M and N be right R-modules. Then N is small simple M-injective if and only if for each small and simple submodule mR of M,*

$$l_N r_R(m) = \mathrm{Hom}_R(M, N)m.$$

Proof. Clearly, $\mathrm{Hom}_R(M,N)m \subset l_N r_R(m)$. Let $x \in l_N r_R(m)$. Define $\varphi : mR \to xR$ by $\varphi(mr) = xr$ for every $r \in R$. Then φ is well-defined because $r_R(m) \subset r_R(x)$. It is clear that φ is an R-homomorphism. Since N is small simple M-injective, there exists an R-homomorphism $\widehat{\varphi} : M \to N$ such that $\widehat{\varphi}\iota_1 = \iota_2\varphi$, where $\iota_1 : mR \to M$ and $\iota_2 : xR \to N$ are the inclusion maps. Hence $x = \varphi(m) = \widehat{\varphi}(m) \in \mathrm{Hom}_R(M,N)m$.

Conversely, let mR be a small and simple submodule of M and let $\varphi : mR \to N$ be an R-homomorphism. Then $\varphi(m) \in l_N r_R(m)$ so by assumption, $\varphi(m) = \widehat{\varphi}(m)$ for some $\widehat{\varphi} \in \mathrm{Hom}_R(M,N)$. This shows that N is small simple M-injective. □

Example 2.3. *Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field, $M_R = R_R$ and $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then N is small simple M-injective.*

Proof. It is clear that only $X = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is the non-zero small and simple submodule of M. Let $\varphi : X \to N$ be R-homomorphism. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in X$, there exists $x_{11}, x_{12} \in F$ such that $\varphi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}$. Then $\varphi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \varphi[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}] = \varphi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}$. It follows that $x_{11} = 0$. Define $\widehat{\varphi} : M \to N$ by $\widehat{\varphi}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} x_{12} & 0 \\ 0 & 0 \end{pmatrix}$. It is clear that $\widehat{\varphi}$ is an R-homomorphism. Then $\widehat{\varphi}(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \widehat{\varphi}[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}] = \widehat{\varphi}(\begin{pmatrix} x_{12} & 0 \\ 0 & 0 \end{pmatrix})\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}$. This shows that $\widehat{\varphi}$ is an extension of φ. Thus N is small simple M-injective. □

Proposition 2.4. *Let M be a right R-module and let $\{N_i : i \in I\}$ be a family of right R-modules. Then the direct product $\prod_{i\in I} N_i$ is small simple M-injective if and only if each N_i is small simple M-injective.*

Proof. $(\Rightarrow)$ Let π_i and φ_i, for each $i \in I$, be the ith projection map and the ith injection map, respectively. We now let $i \in I$, mR a small and simple submodule of M and let $\varphi : mR \to N_i$ be an R-homomorphism. Then by assumption, there exists an R-homomorphism $\widehat{\varphi} : M \to N_i$ such that $\widehat{\varphi}\iota = \varphi_i\varphi$ where $\iota : mR \to M$ is the inclusion map. Thus $\varphi = \pi_i\widehat{\varphi}\iota$.

$(\Leftarrow)$ Let mR be a small and simple submodule of M and let $\varphi : mR \to \prod_{i\in I} N_i$ be an R-homomorphism. Then for each $i \in I$, there exists an R-homomorphism

$\alpha_i : M \to N_i$ such that $\alpha_i\iota = \pi_i\varphi$ where $\iota : mR \to M$ is the inclusion map. Hence we obtain (product) $\widehat{\varphi} : M \to \prod_{i\in I} N_i$ with $\pi_i\widehat{\varphi} = \alpha_i$ and $\pi_i\widehat{\varphi}\iota = \alpha_i\iota$ which implies $\widehat{\varphi}\iota = \varphi$. □

Lemma 2.5. *Let N_i $(1 \le i \le n)$ be small simple M-injective modules. Then $\oplus_{i=1}^{n} N_i$ is small simple M-injective.*

Proof. It is enough to prove the result for $n = 2$. Let mR be a small and simple submodule of M and $\varphi : mR \to N_1 \bigoplus N_2$ be an R-homomorphism. Since N_1 and N_2 are small simple M-injective, there exists R-homomorphisms $\varphi_1 : M \to N_1$ and $\varphi_2 : M \to N_2$ such that $\varphi_1\iota = \pi_1\varphi$ and $\varphi_2\iota = \pi_2\varphi$ where π_1 and π_2 are the projection maps from $N_1 \bigoplus N_2$ to N_1 and N_2, respectively, and $\iota : mR \to M$ is the inclusion map. Put $\widehat{\varphi} = \iota_1\varphi_1 + \iota_2\varphi_2 : M \to N_1 \bigoplus N_2$ where ι_1 and ι_2 are the injection maps from N_1 and N_2 to $N_1 \bigoplus N_2$, respectively. Thus it is clear that $\widehat{\varphi}$ extends φ. □

Lemma 2.6. *Any direct summand of a small simple M-injective module is again small simple M-injective.*

Proof. By definition. □

Theorem 2.7. *The following conditions are equivalent for a projective module M:*

(1) *Every small and simple submodule of M is projective.*

(2) *Every factor module of a small simple M-injective module is small simple M-injective.*

(3) *Every factor module of an injective R-module is small simple M-injective.*

Proof. (1) ⇒ (2) Let N be a small simple M-injective module, X a submodule of N, mR a small and simple submodule of M, and let $\varphi : mR \to N/X$ be an R-homomorphism. Then by (1), there exists an R-homomorphism $\alpha : mR \to N$ such that $\varphi = \eta\alpha$ where $\eta : N \to N/X$ is the natural R-epimorphism. Hence α can be extended to an R-homomorphism $\beta : M \to N$. Then $\eta\beta$ is an extension of φ to M.

(2)⇒ (3) is clear.

(3)⇒(1) Let mR be a small and simple submodule of M, $\alpha : A \to B$ an R-epimorphism, and let $\varphi : mR \to B$ be an R-homomorphism. Embed A in an injective module E [1, 18.6]. Then $B \simeq A/Ker(\alpha)$ is a submodule of $E/Ker(\alpha)$ so by hypothesis, φ can be extended to $\widehat{\varphi} : M \to E/Ker(\alpha)$. Since M is projective, $\widehat{\varphi}$ can be lifted to $\beta : M \to E$. It is clear that $\beta(mR) \subset A$. Therefore we have lifted φ. □

3. The Endomorphism Ring

A right R-module M is called *small simple quasi-injective* if it is small simple M-injective. In this section, we give some characterizations and properties of small simple quasi-injective modules.

Lemma 3.1. *Let M be a right $R-$module and $S = End_R(M)$. Then the following conditions are equivalent:*

(1) *M is small simple quasi-injective.*

(2) *If mR is small and simple, $m \in M$, then $l_M r_R(m) = Sm$.*

(3) *If mR is small and simple and $r_R(m) \subset r_R(n)$, $m, n \in M$, then $Sn \subset Sm$.*

(4) *If mR is small and simple, $m \in M$, then $l_M(r_R(m) \cap aR) = l_M(a) + Sm$ for all $a \in R$.*

(5) *If mR is small and simple, $m \in M$, and $\gamma : mR \to M$ is an R-homomorphism, then $\gamma(m) \in Sm$.*

Proof. (1) $\Leftrightarrow$ (2) by Lemma 2.2.

(2)$\Rightarrow$ (3) If $r_R(m) \subset r_R(n)$, where $m, n \in M$ with mR is small and simple, then $l_M r_R(n) \subset l_M r_R(m)$. Since $Sn \subset l_M r_R(n)$ and by (2), $l_M r_R(m) = Sm$, so we have $Sn \subset Sm$.

(3)$\Rightarrow$(4) Let $a \in R$, $m \in M$ with mR is small and simple and let $x \in l_M(r_R(m) \cap aR)$. Then $x(r(m) \cap aR) = 0$ so $r(ma) \subset r(xa)$. If $ma = 0$, then $mar = 0$ for all $r \in R$ so $xa = 0$. It follows that $x \in l(a) \subset l(a) + Sm$. If $ma \neq 0$, then $maR = mR$ and so $Sxa \subset Sma$ by (3). Thus $xa = \varphi(ma)$, $\varphi \in S$ and hence $(x - \varphi(m)) \in l_M(a)$. It follows that $x \in l_M(a) + Sm$. The other inclusion is clear.

(4)$\Rightarrow$ (2) Put $a = 1_R$.

(3)$\Rightarrow$ (5) Let mR be small and simple, $m \in M$, and let $\gamma : mR \to M$ be an R-homomorphism. Then $r_R(m) \subset r_R(\gamma(m))$ so by (3) we have $S\gamma(m) \subset Sm$. It follows that $\gamma(m) \in Sm$.

(5)$\Rightarrow$ (1) Let mR be a small and simple submodule of M and let $\varphi : mR \to M$ be an $R-$homomorphism. Then by (5), $\varphi(m) \in Sm$. Write $\varphi(m) = \widehat{\varphi}(m)$ where $\widehat{\varphi} \in S$. It is clear that $\widehat{\varphi}$ is an extension of φ. □

Lemma 3.2. *Let M be a small simple quasi-injective module and $S = End_R(M)$. If $m \in M$ and $\alpha \in S$ with $\alpha(M)$ is small and simple, then*

$$l_S(Ker(\alpha) \cap mR) = l_S(m) + S\alpha.$$

Proof. It is always the case that $l_S(m) + S\alpha \subset l_S(Ker(\alpha) \cap mR)$. Let $\beta \in l_S(Ker(\alpha) \cap mR)$. Then $r_R(\alpha(m)) \subset r_R(\beta(m))$, so $l_M r_R(\beta(m)) \subset l_M r_R(\alpha(m))$. Case $\alpha(m) = 0$ is clear. If $\alpha(m) \neq 0$, then $\alpha(m)R$ is simple and small in M, hence $S\beta(m) \subset l_M r_R(\beta(m)) \subset l_M r_R(\alpha(m)) = S\alpha(m)$ by Lemma 3.1, so $\beta(m) = \gamma\alpha(m)$, $\gamma \in S$. It follows that $(\beta - \gamma\alpha) \in l_S(m)$, and hence $\beta \in l_S(m) + S\alpha$. □

Following [9], a right R-module M is called a *principal self-generator* if every element $m \in M$ has the form $m = \gamma(m_1)$ for some $\gamma : M \to mR$.

Proposition 3.3. *Let M be a principal module which is a principal self-generator and let $S = End_R(M)$. Then the following conditions are equivalent:*

(1) *M is small simple quasi-injective.*

(2) *$l_S(Ker(\alpha) \cap mR) = l_S(m) + S\alpha$ for all $m \in M$ and $\alpha \in S$ with $\alpha(M)$ is small and simple in M.*

(3) *$l_S(Ker(\alpha)) = S\alpha$ for all $\alpha \in S$ with $\alpha(M)$ is small and simple in M.*

(4) *$Ker(\alpha) \subset Ker(\beta)$, where $\alpha, \beta \in S$ with $\alpha(M)$ is small and simple in M, implies $S\beta \subset S\alpha$.*

Proof. (1) $\Rightarrow$ (2) by Lemma 3.2.

(2)$\Rightarrow$ (3) If $M = m_0R$, take $m = m_0$ in (2).

(3) $\Rightarrow$ (4) If $Ker(\alpha) \subset Ker(\beta)$, then $l_S(Ker(\beta)) \subset l_S(Ker(\alpha)$. It follows that $S\beta \subset l_S(Ker(\beta)) \subset l_S(Ker(\alpha) = S\alpha$.

(4) $\Rightarrow$ (1) Let mR be a small and simple submodule of M and $\varphi : mR \to M$ be an R-homomorphism. Since M is a principal self-generator, there exists $\beta \in S$ such that $\beta(m_1) = m$, so $Ker(\beta) \subset Ker(\varphi\beta)$ and $\beta(M)$ is small and simple in M. Then by (4), $S\varphi\beta \subset S\beta$, write $\varphi\beta = \widehat{\varphi}\beta$, $\widehat{\varphi} \in S$. This shows that $\widehat{\varphi}$ extends φ. □

Theorem 3.4. *Let M be a small simple quasi-injective module, $m, n \in M$ and mR is small and simple.*

(1) *If mR embeds in nR, then Sm is an image of Sn.*

(2) *If nR is an image of mR, then Sn embeds in Sm.*

(3) *If $mR \simeq nR$, then $Sm \simeq Sn$.*

Proof. (1) Let $f : mR \to nR$ be an R–monomorphism. Let $\iota_1 : mR \to M$ and $\iota_2 : nR \to M$ be the inclusion maps. Since M is small simple quasi-injective, there exists an R–homomorphism $\widehat{f} : M \to M$ such that $\iota_2 f = \widehat{f}\iota_1$. Let $\sigma : Sn \to Sm$ defined by $\sigma(\alpha(n)) = \alpha\widehat{f}(m)$ for every $\alpha \in S$. Since $\sigma(\alpha(n)) = \alpha f(m) \in \alpha(nR)$, σ is well-defined. It is clear that σ is an S–homomorphism. Note that $f(m)R$ is simple and $f(m)R = \widehat{f}(m)R \ll M$ by [1, Lemma 5.18]. Since f is monic, $r_R(f(m)) = r_R(m)$ and hence by Lemma 3.1, $Sm \subset Sf(m)$. Then $m \in Sf(m) \subset \sigma(Sn)$.

(2) By the same notations as in (1), let $f : mR \to nR$ be an R–epimorphism. Write $f(ms) = n$, $s \in R$. Since M is small simple quasi-injective, f can be extended to $\widehat{f} : M \to M$ such that $\iota_2 f = \widehat{f}\iota_1$. Define $\sigma : Sn \to Sm$ by $\sigma(\alpha(n)) = \alpha\widehat{f}(ms)$ for every $\alpha \in S$. It is clear that σ is S-homomorphism. If $\alpha(n) \in Ker(\sigma)$, then $0 = \sigma(\alpha(n)) = \alpha\widehat{f}(ms) = \alpha f(ms) = \alpha(n)$. This shows that σ is an S–monomorphism.

(3) Follows from (1) and (2). □

Proposition 3.5. *Let M be a principal module which is a principal self-generator. If M is small simple quasi-injective, then $Soc(M_R) \subset r_M(J(S))$.*

Proof. Let mR be a simple submodule of M. Suppose $\alpha(m) \neq 0$ for some $\alpha \in J(S)$. As M is a principal self-generator, $mR = \sum_{s \in I} s(M)$ for some $I \subset S$. Since mR is a simple, $mR = s(M)$ for some $0 \neq s \in I$. Then $\alpha s \neq 0$ and $Ker(\alpha s) = Ker(s)$. Note that $\alpha s(M)$ is a nonzero homomorphic image of the simple $s(M)$, then $\alpha s(M)$ is simple. Since M is a principal module, $J(M) \ll M$ so we have $J(S)M \subset J(M)$, it follows that $\alpha s(M)$ is a small submodule of M. Since M is small simple quasi-injective, $l_S(ker(\alpha s)) = S\alpha s$. Thus $l_S(ker(s)) = S\alpha s$. Write $s = \beta\alpha s$ where $\beta \in S$. Then $(1-\beta\alpha)s = 0$ and so $s = (1-\beta\alpha)^{-1}0$. It follows that $s = 0$, a contradiction. □

Let M be a right $R-$module with $S = \mathrm{End}_R(M)$. Following [6], write $\Delta = \{s \in S : ker(s) \subset^e M\}$. It is known that Δ is an ideal of S [6, Lemma 3.2].

Proposition 3.6. *Let M be a principal module which is a principal self-generator and $Soc(M_R) \subset^e M$. If M is small simple quasi-injective, then $J(S) \subset \Delta$.*

Proof. Let $s \in J(S)$. If $Ker(s) \not\subset^e M$, then $Ker(s) \cap N = 0$ for some nonzero submodule N of M. Since $Soc(M_R) \subset^e M$, $Soc(M_R) \cap N \neq 0$. Then there exists a simple submodule mR of M such that $mR \subset Soc(M_R) \cap N$ [1, Corollary 9.10]. As M is a principal self-generator and mR is simple, $mR = t(M)$ for some $t \in S$. It follows that $Ker(st) = Ker(t)$. Since $st(M)$ is a nonzero homomorphic image of the simple $t(M)$, $st(M) = t(M)$. It is clear that $st(M) \ll M$. Then $t \in l_S(ker(t)) = l_S(ker(st) = Sst$. Write $t = \alpha st$ where $\alpha \in S$. It follows that $t = (1-\alpha s)^{-1}0$. Then $t = 0$, a contradiction. □

Proposition 3.7. *Let M be a principal nonsingular module which is a principal self-generator and $Soc(M_R) \subset^e M$. If M is small simple quasi-injective, then $J(S) = 0$.*

Proof. Since $J(S) \subset \Delta$ by Proposition 3.6, we show that $\Delta = 0$. Let $s \in \Delta$ and let $m \in M$. Define $\varphi : R \to M$ by $\varphi(r) = mr$ for every $r \in R$. It is clear that φ is an $R-$homomorphism. Thus

$$\begin{aligned} r_R(s(m)) &= \{r \in R : s(mr) = 0\} \\ &= \{r \in R : mr \in Ker(s)\} \\ &= \{r \in R : \varphi(r) \in Ker(s)\} \\ &= \varphi^{-1}(Ker(s)). \end{aligned}$$

It follows that $\varphi^{-1}(Ker(s)) \subset^e R$ [4, Lemma 5.8(a)] so $r_R(s(m)) \subset^e R$. Thus $s(m) \in Z(M_R) = 0$ because M is nonsingular. As this is true for all $m \in M$, we have $s = 0$. Hence $\Delta = 0$ as required. □

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