OPTION PRICING FOR JUMP - DIFFUSION WITH STOCHASTIC VOLATILITY AND INTENSITY

MONTAKAN THONGPAN

A THESIS SUBMITTED IN PARTIAL FULLFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE PROGRAM IN MATHEMATICS FACULTY OF SCIENCE AND TECHNOLOGY RAJAMANGALA UNIVERSITY OF TECHNOLOGY THANYABURI

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Thesis Title	Option Pricing for Jump - Diffusion with Stochastic Volatility and Intensity
Name - Surname	Miss Montakan Thongpan
Program	Mathematics
Thesis Advisor	Assistant Professor Sarun Wongwai, Ph.D.
Academic Year	2013

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ABSTRACT

It is widely believed that the volatility of asset returns tends to be time varying and occasionally clustered, which leads to various stochastic volatility models. The assumption of constant intensity is relaxed to allow stochastic intensity. Combinations lead to stochastic volatility and stochastic intensity models, as well as jump-diffusion with stochastic volatility and stochastic intensity models.

In this thesis, a jump-diffusion combined with stochastic volatility model and stochastic intensity is considered and its presentations include: the dynamics of asset price in which the asset price follows a geometric Brownian motion, compound Poisson processes with the stochastic volatility following Heston model, and the stochastic intensity following mean reverting process.

A formula of the European option is calculated by using a technique based on the characteristic function of the underlined asset which can be expressed in an explicit formula.

Keywords: jump - diffusion model, stochastic volatility, intensity, characteristic functions.

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Montakan Thongpan

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CHAPTER 1
INTRODUCTION

A financial derivative is a financial instrument. The value of derivative determined by the price of something else that called the underlying. Example: Options, Futures, Swap. A call option gives the right to buy the assets whereas a put option gives the right to sell the asset at a strike price. European options can only be exercised at expiration date. American option can be exercised any time during the life of the option. The problem of pricing the option and modeling of the underlying assets. How much should the buyer pay for the option? How do we model the underlying asset specific on a stock price?

In 1973, Fischer Black and Myron Scholes explain if option are correctly priced in the market, it should not be possible to make sure profits by creating portfolios of long and short positions in option and their underlying stocks. Using this principle, a theoretical valuation formula for options is derived.

In 1993, Heston use a new technique to derive a closed – form solution for the price of a European call option on an asset with stochastic volatility. The model allows arbitrary correlation between volatility and spot asset returns.

In 1996, Bates developed for pricing American potions on stochastic volatility jump - diffusion processes under systematic jump and volatility risk. We are interested in seeing to what extent the Bates model.

1.1 Purpose of the Study

1.1.1 To investigate the option pricing for jump - diffusion with stochastic volatility and intensity.

1.1.2 To find a closed - form solution for European call option of jump - diffusion with stochastic volatility and intensity.

1.2 Theoretical Perspective

1.2.1 Basic Stochastic Processes

1.2.2 Elementary Stochastic Calculus

1.3 Delimitations and Limitations of the Study

For this thesis, we have the scopes and the limitations of studying which are concerned to the previous works which are:

1.3.1 To investigate the option pricing for jump - diffusion with stochastic volatility and intensity.

1.3.2 To find a closed-form solution for European call option of jump - diffusion with stochastic volatility and intensity.

1.4 Significance of the Study

Option pricing for jump - diffusion with stochastic volatility and intensity is presented by using stochastic calculus.

CHAPTER 2
LITERATURE REVIEW

In this chapter we give notations, definitions and fundamental theories of the jump-diffusion model, stochastic volatility, intensity, characteristic functions which are used in this thesis.

2.1 Option

2.1.1 call option is a contract to buy at a specified future time a certain amount of an underlying asset at a specified price.

2.1.2 put option is a contract to sell at a specified future time a certain amount of an underlying asset at a specified price.

According to terms on exercise in the contract, options have the following types:

- European options can be exercised only on the expiration date.
- American options can be exercised on or prior to the expiration date.

Define K and T are strike price and expiration date respectively, then an option's payoff (value) $C(T,\ \mathrm{S})$ at expiration date is:

$$(S_T - \mathrm{K})^+ = \max(S_T - \mathrm{K}, 0) \qquad \text{(call option)}$$

$$(\mathrm{K} - S_T)^+ = \max(\mathrm{K} - S_T, 0) \qquad \text{(put option)}$$

where S_T denotes the price of the underlying asset at the expiration date $t = T$. Option is a contingent claim. Take a call option as example. If S_T , the underlying asset's price at expiration date, is higher than the strike price K, then the holder of the option can exercise the rights to buy the asset at the strike price K (to gain profits). Otherwise, the option is a worthless. That is

$$C(T, S_T) = \begin{cases} S_T - K & if\ \ S_T > K \\ \\ 0 & otherwise. \end{cases}$$

In the case of $S_T > K$ the option is called "in the money". It is said to be "out of the money" if $S_T < K$. If $S_T = K$, it is "at the money". Similarly, the payoff function is $(K - S_T)^+$ for a European put option.

The price paid for a contingent claim is called the premium. When the option is traded on an organized market, the premium is quoted by the market. Otherwise, the problem is to price the option. Also, even if the option is traded on an organized market, it can be interesting to detect some possible abnormalities in the market.

Taking into account the premium, the total gain of the option holder at its expiration date is[Total gain] = [Gain of the option at expiration] - [Premium] i.e.,

Total gain = $(S_T - K)^+$ - premium (call option)

Total gain = $(K - S_T)^+$ - premium (put option)

As a derived security, the price of an option varies with the price of its underlying asset. Since the underlying asset is a risky asset, its price is a random variable.

2.2 Stochastic Process

Definition. [11] A stochastic process X is a collection of random variables

$$(X_t, t \in T) = (X_t(\omega), t \in T, \omega \in \Omega),$$

defined on some space Ω.

2.3 Brownian Motion

Definition. [11] A stochastic process $W = (W_t, t \in [0, \infty))$ is called standard Brownian motion or a Wiener process if the following conditions are satisfied:

(1) It starts at zero : $W_0 = 0$.

(2) For every $t > 0$, W_t has a normal $N(0,t)$ distribution.

(3) It has continuous sample paths: "no jumps"

2.4 Itô Formula [15]

Suppose that $F(t,x)$ is a real – valued function with continuous partial derivative $F_t(t,x)$, $F_x(t,x)$ and $F_{xx}(t,x)$ for all $t \geq 0$ and $x \in \mathrm{R}$. Then $F(t,W_t)$ is an Itô process such that

$$F(T,W_T)-F(0,W_0)=\int_0^T\left(F_t(t,W_t)+\frac{1}{2}F_{xx}(t,W_t)\right)dt+\int_0^T F_x\,(t,W_t)dW_t.$$

In differential notation this formula can be written as

$$dF(t,W_t)=\left(F_t\ (t,W_t)+\frac{1}{2}F_{xx}(t,W_t)\right)dt+F_x\ (t,W_t)dW_t$$

where

$$dt\cdot dt=dW_t\cdot dt=dt\cdot dW_t=0$$

and

$$dW_t\cdot dW_t=\left(dW_t\right)^2=dt.$$

2.5 Poisson Process

Definition. [11] A stochastic process $(N_t, t\in[0,\infty))$ is called an homogeneous Poisson process or simply a Poisson process with intensity or rate $\lambda>0$ if the following conditions are satisfied:

(1) It starts at zero : $N_0=0.$

(2) It has stationary, independent increments.

(3) For every $t>0,\ N_t$ has a Poisson Poi (λt) distribution.

2.6 Probability

2.6.1 Definition. [11] The collection of the probabilities

$$F_X(x)=P(X\leq x)=P(\{\omega : X(\omega)\leq x\}),\ x\in\mathrm{R}=(-\infty,\infty),$$

is the distribution function F_X of X.

2.6.2 Definition. [11] Most continuous distributions of interest have a density f_X :

$$F_X(x)=\int_{-\infty}^{x} f_X(y)dy,\ x\in\mathrm{R},$$

where

$$f_X(x) \geq 0 \text{ for every } x \in \mathrm{R} \text{ and } \int_{-\infty}^{\infty} f_X(y)dy = 1.$$

An important continuous distribution is the normal or Gaussian distribution $N(\mu,\sigma^2)$ with parameters $\mu \in \mathrm{R}, \sigma^2 > 0.$ It has density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},\ x \in \mathrm{R}.$$

2.7 Random Variable

Theorem. [13] Let X be a random variable and let $Z = g(x)$ for some function g.

Suppose X is continuous with probability density function (pdf) $f_X(x)$.

If $\int_{-\infty}^{\infty} |g(x)| f_X(x)dx < \infty$, then the expectation of Z exists and it is given by

$$\mathrm{E}[\mathrm{Z}] = \int_{-\infty}^{\infty} g(x) f_X(x)dx.$$

2.8 The Normal Distribution [13]

We will proceed by first introducing for applications and for statistical inference, in through it the general normal distribution.

Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz. \tag{2.1}$$

The integral exists because the integrable function, that is,

$$0 < \exp\left(\frac{-z^2}{2}\right) < \exp\left(-|z|+1\right),\ -\infty < z < \infty,$$

and

$$\int_{-\infty}^{\infty} \exp\left(-|z|+1\right) dz = 2e.$$

To evaluate the integral I, we note that $I > 0$ and that I^2 may be written

$$I^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{z^2+w^2}{2}\right)dzdw.$$

This iterated integral can be evaluated by changing to polar coordinates. If we set $z = r\cos\theta$ and $w = r\sin\theta$, we have

$$\begin{aligned} I^2 &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} r\, dr\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \\ &= 1. \end{aligned}$$

Because the integrand of display (2.1) is positive on R and integrates to 1 over R, it is a pdf. of a continuous random variable with support R. We denote this random variable by Z. In summary, Z has the pdf.,

$$f(z) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-z^2}{2}\right), \quad -\infty < z < \infty.$$

Note [13] X has $N(\mu,\sigma^2)$ distribution if and only if $Z = \dfrac{x-\mu}{\sigma}$ has a $N(0,1)$ distribution.

2.9 The Stock Price Process [14]

It is generally assumed that stock price follow geometric Brownian motion under the real world measure P,

$$dS_t = S_t\mu dt + S_t\sigma dW_t \tag{2.2}$$

where $\mu \in \mathrm{R}$ and $S_t, \sigma \in \mathrm{R}^+$, W_t is Brownian motion and the process is defined on $[0,T]$.Equation (2.2) is known as Black – Scholes model or diffusion model.

A solution S_t, to this equation can be found with the help of Ito's formula. Let $f(t,x) = \ln(x)$. It follows from that $f(t,x) \in C^2([0,\infty)\times \mathrm{R})$. Fortunately, if we assume that $S_t \in \mathrm{R}^+$, we can define $f(t,x) \in C^2([0,\infty)\times \mathrm{R}^+)$. We have

$$d\ln(S_t) = \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}\left(dS_t\right)^2.$$

By Itô Formula,

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dS_t)^2 \\
&= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(S_t\mu dt + S_t\sigma dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(S_t\mu dt + S_t\sigma dW_t)^2 \\
&= 0 + \frac{1}{S_t}(S_t\mu dt + S_t\sigma dW_t) - \frac{1}{2S_t^2}(S_t\mu dt + S_t\sigma dW_t)^2 \\
&= \frac{1}{S_t}(S_t\mu dt + S_t\sigma dW_t) - \frac{1}{2S_t^2}(S_t\mu dt + S_t\sigma dW_t)^2 \\
&= \frac{1}{S_t}(S_t\mu dt + S_t\sigma dW_t) - \frac{1}{2S_t^2}[(S_t\mu dt)^2 + 2S_t\mu S_t\sigma dW_t dt + (S_t\sigma dW_t)^2] \\
&= (\frac{1}{S_t}S_t\mu dt + \frac{1}{S_t}S_t\sigma dW_t) - \frac{1}{2S_t^2}\left[(S_t{}^2\mu^2(dt)^2 + 2S_t\mu S_t\sigma dW_t dt + S_t{}^2\sigma^2(dW_t)^2\right] \\
&= \mu dt + \sigma dW_t - \frac{1}{2}\left[\mu^2(dt)^2 + 2\mu\sigma dt dW_t + \sigma^2(dW_t)^2\right] \\
&= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\
&= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t
\end{aligned}
$$

which integral notation is

$$d\ln(S_t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

$$\int_0^t d\ln(S_u) = \int_0^t(\mu - \frac{1}{2}\sigma^2)du + \int_0^t \sigma dW_u$$

$$
\begin{aligned}
\ln(S_t) - \ln(S_0) &= \int_0^t(\mu - \frac{1}{2}\sigma^2)du + \int_0^t \sigma dW_u \\
&= (\mu - \frac{1}{2}\sigma^2)\int_0^t du + \sigma\int_0^t dW_u \\
&= (\mu - \frac{1}{2}\sigma^2)u\Big|_{u=0}^{t} + \sigma W_u\Big|_{u=0}^{t} \\
&= (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t. \qquad (2.3)
\end{aligned}
$$

The solution S_t is

$$\ln\left(\frac{S_t}{S_0}\right)=(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t$$

$$e^{\ln\left(\frac{S_t}{S_0}\right)}=e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

$$\frac{S_t}{S_0}=e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

$$S_t=S_0e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

$$S_t=S_0\exp((\mu-\frac{1}{2}\sigma^2)t+\sigma W_t).$$

Thus by assuming that the stock price follow the geometric Brownian motion described in equation (2.2), we are also assuming that the stock price is lognormally distributed. There are ample empirical evidence to support this assumption. This means that from equation (2.3)

$$\ln(S_t)\sim N\left(\ln\left(S_0\right)+\left(\mu-\frac{1}{2}\sigma^2\right)t,\ \sigma^2 t\right).$$

The next aim is to find a probability measure under which $\tilde{S}_t=\frac{S_t}{B_t}$ is a martingale , called the risk – neutral probability measure. The discounted process

$$\begin{aligned}\tilde{S}_t&=\frac{S_t}{B_t}\\&=e^{-rt}S_0\,\mathrm{e}^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}\\&=S_0\,\mathrm{e}^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t-rt}\\&=S_0\,\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t+\sigma W_t}.\end{aligned}$$

That is,

$$\tilde{S}_t=S_0\exp((\mu-r-\frac{1}{2}\sigma^2)t+\sigma W_t)$$

where $B_t=e^{rt}$ and r is the constant risk-free rate of interest.

To get the stochastic process driving $\tilde{S}_t = S_t e^{-rt}$, we again use Ito's formula

$$
\begin{aligned}
df(t,S_t) &= d\tilde{S}_t \\
&= d(S_t e^{-rt}) \\
&= S_t de^{-rt} + e^{-rt} dS_t \\
&= S_t (e^{-rt}) d(-rt) + e^{-rt}(S_t \mu dt + S_t \sigma dW_t) \\
&= S_t (e^{-rt})(-r)dt + e^{-rt}(S_t \mu dt + e^{-rt} S_t \sigma dW_t) \\
&= -rS_t e^{-rt} dt + e^{-rt} S_t \mu dt + e^{-rt} S_t \sigma dW_t \\
&= (\mu - r) S_t e^{-rt} dt + e^{-rt} S_t \sigma dW_t \\
&= (\mu - r)\tilde{S}_t dt + \tilde{S}_t \sigma dW_t .
\end{aligned}
$$

Thus

$$d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \tilde{S}_t \sigma dW_t .$$

2.10 Girsanov's theorem [14]

Girsanov's theorem is used to transform stochastic processes in terms of their drift parameters. In option pricing, Girsanov's theorem is used to find a probability measure under which the risk-free rate adjusted stock price process is a martingale.

2.10.1 Definition. Let $v = v(S,T)$ be the class of functions

$$f(t,\omega):[0,\infty)\times\Omega \to \mathrm{R}$$

such that

1. $(t,\omega) \to f(t,\omega)$ is $B \times F$ -measurable, where B is the Borel sets on $[0,\infty)$.
2. $f(t,\omega)$ is adapted.
3. $E\left[\int_0^T f(t,\omega)^2 dt\right] < \infty.$

2.10.2 Theorem. Girsanov's theorem. Let $X_t \in \mathrm{R}$ be an Ito process, of the form

$$dX_t = \beta(t,\omega)dt + \theta(t,\omega)dW_t$$

with $t \le T < \infty$. Suppose that there exist a $v(t,\omega)$-process $u(t,\omega) \in \mathrm{R}$ and $\alpha(t,\omega) \in \mathrm{R}$ such that

$$\theta(t,\omega)u(t,\omega) = \beta(t,\omega) - \alpha(t,\omega).$$

Since we are only looking at the on dimensional case,

$$u(t,\omega) = \frac{\beta(t,\omega) - \alpha(t,\omega)}{\theta(t,\omega)}.$$

We further assume that

$$E_P\left[\exp\left(\frac{1}{2}\int_0^T u^2(s,\omega)ds\right)\right] < \infty .$$

Let

$$M_t = \exp\left(-\int_0^t u(s,\omega)\,dW_s - \int_0^t u^2(s,\omega)\,ds\right)$$

and

$$dQ = M_T dP .$$

We then have that

$$\tilde{W}_t = W_t + \int_0^t u(s,\omega)ds$$

is a Brownian motion with respect to Q. X_t in terms of $\tilde{W}_t$ is

$$dX_t = \alpha(t,\omega) + \theta(t,\omega)d\tilde{W}_t .$$

Ito's clear that the process $\tilde{S}_t$ has a trend, $(\mu - r)\tilde{S}_t$. This trend causes $\tilde{S}_t$ not to be a P-martingale (a martingale under probability measure P).

The risk - neutral probability measure is found by employing Girsanov's theorem. By using the notation of the Girsanov theorem we can define, for the process $\tilde{S}_t$.

$$\begin{aligned} u(t,\omega) &= \frac{(\mu - r)\tilde{S}_t}{\sigma \tilde{S}_t} \\ &= \frac{(\mu - r)}{\sigma}. \end{aligned}$$

Note that $\alpha(t,\omega) \equiv 0$ and $u(t,\omega) = u$ is a finite scalar since we assumed that σ is strictly positive. The result of this is that is met and $u \in \nu(t,\omega)$.

Since

$$E_P\left[\exp(\frac{1}{2}\int_0^T u^2(s,\omega)ds)\right] = E_P\left[\exp\left(\frac{1}{2}\int_0^T\left(\frac{\mu - r}{\sigma}\right)^2 ds\right)\right] < \infty, \tag{2.4}$$

M_t was defined in Theorem 2.10.2 as follows.

$$M_t = \exp\left(-\int_0^t u(s,\omega)dW_s - \int_0^t u^2(s,\omega)ds\right). \tag{2.5}$$

In this case, for $u(t,\omega) = u$

$$\begin{aligned} M_t &= \exp\left(-\int_0^t udW_s - \int_0^t u^2 ds\right) \\ &= \mathrm{e}^{-uW_s\Big|_{s=0}^{t} - u^2 s\Big|_{s=0}^{t}} \\ &= \mathrm{e}^{-u[W_t - W_0] - u^2[t-0]} \\ &= \mathrm{e}^{-uW_t - u^2 t}. \end{aligned}$$

The new measure, the risk- neutral probability measure can be defined as

$$dQ = M_T dP\,.$$

We can define a new process

$$\begin{aligned} \tilde{W}_t &= W_t + \int_0^t u(s,\omega)ds \\ &= W_t + \int_0^t uds \\ &= W_t + us\Big|_{s=0}^{t} \\ &= W_t + u[t-0] \\ &= W_t + ut \end{aligned}$$

which is a Q – Brownian motion. The original process $\tilde{S}_t$, in terms of $\tilde{W}_t$ is

$$\begin{aligned} d\tilde{S}_t &= (\mu - r)\tilde{S}_t dt + \sigma\tilde{S}_t dW_t \\ &= (\mu - r)\tilde{S}_t dt + \sigma\tilde{S}_t d(\tilde{W}_t - ut) \end{aligned}$$

$$
\begin{aligned}
d\tilde{S}_t &= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t (d\tilde{W}_t - udt) \\
&= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t d\tilde{W}_t - \sigma \tilde{S}_t udt \\
&= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t d\tilde{W}_t - \sigma \tilde{S}_t \frac{(\mu - r)}{\sigma} dt \\
&= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t d\tilde{W}_t - \tilde{S}_t (\mu - r) dt \\
&= \sigma \tilde{S}_t d\tilde{W}_t .
\end{aligned}
$$

The scalar $u(t,s) = \dfrac{\mu - r}{\sigma}$ is also known as the market price of risk. If $\mu = r$, then the investor is called risk – neutral and $dQ = dP$.

Since

$$E_P\left[\exp\left(\frac{1}{2}\int_0^T u^2(s,\omega)ds\right)\right] = E_P\left[\exp\left(\frac{1}{2}\int_0^T \left(\frac{\mu - r}{\sigma}\right)^2 ds\right)\right] < \infty$$

and

$$M_t = \exp\left(-\int_0^t u(s,\omega)dW_s - \int_0^t u^2(s,\omega)ds\right).$$

In this case, for $u(t,\omega) = 0$

$$
\begin{aligned}
M_t &= \exp\left(-\int_0^t udW_s - \int_0^t u^2 ds\right) \\
&= e^0 \\
&= 1
\end{aligned}
$$

and

$$dQ = M_T dP .$$

We have

$$dQ = dP .$$

Under the measure Q we price instruments as if they are risk- neutral.

2.11 Heston model [7]

The Heston model assumes that S_t, the price of the asset, is determined by a stochastic process:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^S$$

where $\mu > 0$, v_t the instantaneous variance is a CIR process:

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v$$

where $\kappa > 0, \theta > 0, \sigma > 0$ and W_t^S, W_t^v are Brownian motion with correlation ρ.

2.12 Bates model [1]

Bates introduced an efficient method is developed for pricing American options on stochastic volatility jump-diffusion processes under systematic jump and volatility risk. The exchange rate S_t satisfy the following process:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^S + kdN_t$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v$$

where k is the random percentage jump conditional on a jump occurring and N_t is a Poisson process with constant intensity λ.

2.13 Feynman-Kac Formula [12]

Let a, b and g be smooth, bounded functions. Let X solve the stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

and let

$$u(x,t) = E\left[g(X_T) \middle| X_t = x\right].$$

Then u is a solution of

$$u_t + au_x + \frac{1}{2}b^2 u_{xx} = 0$$

$$u(x,T) = g(x)$$

for $t < T$.

CHAPTER 3
RESEARCH RESULT

In this chapter, we present the results of jump - diffusion with stochastic volatility and intensity.

3.1 Option Pricing for Jump-Diffusion with Stochastic Volatility and Intensity

The propose model assumes that the underlying asset has the following dynamics under risk-neutral measure,

$$\left.\begin{aligned} \frac{dS_t}{S_t} &= (r-\lambda_t m)dt + \sqrt{v_t}\,dW_t^S + Y_t dN_t \\ dv_t &= \kappa_v(\theta_v - v_t)dt + \sigma\sqrt{v_t}\,dW_t^v \\ d\lambda_t &= \kappa_\lambda(\theta_\lambda - \lambda_t)dt + \varepsilon\sqrt{v_t}\,dW_t^\lambda \end{aligned}\right\} \tag{3.1.1}$$

where S_t is the price of the stock, r is the risk free rate, m is the expected of Y_t, v_t is the instantaneous variance, κ_v is the rate at which v_t reverts to θ_v, θ_v is the long variance, σ is the volatility of the stock's returns, Y_t is jump size with normal distribution, N_t is a Poisson counter with intensity λ, W_t^S and W_t^v are Brownian motion with correlation ρ. For the intensity dynamics we have: κ_λ is a mean-reverting rate, θ_λ is the long term intensity, ε is a volatility of jump intensity and W_t^λ is a standard Brownian motion. We assume that jump process are independent of W_t^S, W_t^v and W_t^λ. A standard Brownian motion W_t^λ is independent of W_t^S and W_t^v.

Denote the characteristic function for $X_T = \ln S_T$ as

$$f(l,v,\lambda,t;x) = E[e^{ixX_T} \mid X_t = l, v_t = v] \tag{3.1.2}$$

where $0 \le t \le T$ and $i = \sqrt{-1}$. Then, the following theorem holds.

Theorem 3.1 Suppose that S_t follows the dynamics in (3.1.1). Then the characteristic function for X_T defined in (3.1.2) is given by

$$f(l,v,\lambda,t;x)=\exp(ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda)$$

where $A(\tau)=-\dfrac{2\kappa_v\theta_v}{\sigma^2}\ln\left[\dfrac{r_2e^{-\frac{1}{2}r_1\tau}+r_1e^{\frac{1}{2}r_2\tau}}{2H}\right]-\dfrac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\left[\dfrac{q_1e^{-\frac{1}{2}q_2\tau}+q_2e^{\frac{1}{2}q_1\tau}}{2E}\right]$,

$$B(\tau)=(u^2-u)\left(\frac{1-e^{-H\tau}}{r_1+r_2e^{-H\tau}}\right),\quad C(\tau)=2F\left[\frac{1-e^{-E\tau}}{q_1e^{-E\tau}+q_2}\right],\quad u=ix$$

$$r_1=(\kappa_v-\rho\sigma u)+H\,,\quad r_2=-(\kappa_v-\rho\sigma u)+H\,,\quad H=\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}$$

$$q_1=\kappa_\lambda+E\,,\quad q_2=-\kappa_\lambda+E\,,\quad E=\sqrt{\kappa_\lambda^2-2\varepsilon^2F}\;,\quad F=-mu+\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy$$

and $\phi_Y(y)$ is a density of random jump size Y_t.

Proof Feynman-Kac formula gives the following PDE for the characteristic function

$$(r-\frac{1}{2}v-\lambda m)f_l+\frac{1}{2}vf_{ll}+\kappa_v(\theta_v-v)f_v+\frac{1}{2}\sigma^2vf_{vv}+\rho\sigma vf_{lv}+\kappa_\lambda(\theta_\lambda-v)f_\lambda$$

$$+\frac{1}{2}\varepsilon^2\lambda f_{\lambda\lambda}+\lambda\int_{-\infty}^{\infty}[f(l+y,v,\lambda,t;\phi)-f(l,v,\lambda,t;\phi)]\phi_Y(y)dy+f_t=0\,, \tag{3.1.3}$$

$$f(l,v,\lambda,T;x)=e^{ixl}\,.$$

Consider for the characteristic function:

$$f(l,v,\lambda,t;x)=\exp(ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda) \tag{3.1.4}$$

where $\tau=T-t$ and $A(\tau=0)=B(\tau=0)=C(\tau=0)$.

We plan to substitute equation (3.1.4) into equation (3.1.3). Firstly, we compute

$$f_l,\,f_{ll},\,f_v,\,f_{vv},\,f_{lv},\,f_\lambda,\,f_{\lambda\lambda},\,f_t\,.$$

That is

$$\begin{aligned}f_l&=\frac{\partial}{\partial l}(e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda})\\&=e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}\frac{\partial}{\partial l}(ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda)\\&=e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}(ix)\end{aligned}$$

$$f_l = ixe^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}$$
$$= ixf \qquad (3.1.4.1)$$

$$f_{ll} = \frac{\partial}{\partial l} ixf$$
$$= \frac{\partial}{\partial l} ixe^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}$$
$$= ix\frac{\partial}{\partial l} e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}$$
$$= ix(ixf)$$
$$= i^2x^2 f$$
$$= -x^2 f\ , \qquad (3.1.4.2)$$

$$f_v = e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}\frac{\partial}{\partial v}(ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda)$$
$$= B(\tau)e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}$$
$$= B(\tau)f\ , \qquad (3.1.4.3)$$

$$f_{vv} = \frac{\partial}{\partial v} B(\tau)f$$
$$= \frac{\partial}{\partial v} B(\tau)e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}$$
$$= B(\tau)(B(\tau)f)$$
$$= B^2(\tau)f\ , \qquad (3.1.4.4)$$

$$f_{lv} = \frac{\partial}{\partial v} ixe^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}$$
$$= ixe^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}\frac{\partial}{\partial v}(ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda)$$
$$= ixB(\tau)e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}$$

$$f_{lv} = ixB(\tau) f \ , \tag{3.1.4.5}$$

$$\begin{aligned} f_{\lambda} &= e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda} \frac{\partial}{\partial \lambda}(ixl + ixr\tau + A(\tau) + B(\tau)v + C(\tau)\lambda) \\ &= C(\tau) e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda} \\ &= C(\tau) f \quad , \end{aligned} \tag{3.1.4.6}$$

$$\begin{aligned} f_{\lambda\lambda} &= \frac{\partial}{\partial \lambda} C(\tau) f \\ &= \frac{\partial}{\partial \lambda} C(\tau) e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda} \\ &= C(\tau) \frac{\partial}{\partial \lambda} e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda} \\ &= C(\tau)[C(\tau) f] \\ &= C^2(\tau) f \ , \end{aligned} \tag{3.1.4.7}$$

$$\begin{aligned} f_t &= e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda} \frac{\partial}{\partial t}(ixl + ixr\tau + A(\tau) + B(\tau)v + C(\tau)\lambda) \\ &= (ixr\frac{\partial \tau}{\partial t} + \frac{\partial A(\tau)}{\partial t} + \frac{\partial B(\tau)v}{\partial t} + \frac{\partial C(\tau)\lambda}{\partial t}) f \end{aligned} \tag{3.1.4.8}$$

Consider the following

$$\begin{aligned} \frac{\partial}{\partial t} A(\tau) &= \frac{\partial A(\tau)}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} \\ &= A_{\tau} \frac{\partial (T-t)}{\partial t} \\ &= -A_{\tau} \ , \end{aligned} \tag{3.1.4.9}$$

$$
\begin{aligned}
\frac{\partial}{\partial t} B(\tau)v &= v\frac{\partial}{\partial t} B(\tau) \\
&= v\left(\frac{\partial B(\tau)}{\partial \tau}\cdot\frac{\partial \tau}{\partial t}\right) \\
&= vB_{\tau}\,\frac{\partial (T-t)}{\partial t} \\
&= -B_{\tau}v\,, \qquad (3.1.4.10)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t} C(\tau)\lambda &= \lambda\frac{\partial}{\partial t} C(\tau) \\
&= \lambda\left(\frac{\partial C(\tau)}{\partial \tau}\cdot\frac{\partial \tau}{\partial t}\right) \\
&= \lambda C_{\tau}\,\frac{\partial (T-t)}{\partial t} \\
&= -C_{\tau}\lambda\,, \qquad (3.1.4.11)
\end{aligned}
$$

Thus

$$f_t = (-ixr - A_{\tau} - B_{\tau}v - C_{\tau}\lambda) f \qquad (3.1.4.12)$$

Consider

$$f(l+y,v,\lambda,t;x) = e.^{ix(l+y)+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}$$

We have

$$
\begin{aligned}
f(l+y,v,\lambda,t;x) - f(l,v,\lambda,t;x) &= e^{ix(l+y)+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda} - e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda} \\
&= e^{ixl+ixy+ixr\iota+A(\tau)+B(\tau)v+C(\tau)\lambda} - e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda} \\
&= e^{ixy}\cdot e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda} - e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda} \\
&= e^{ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda}(e^{ixy}-1) \\
&= (e^{ixy}-1) f. \qquad (3.1.4.13)
\end{aligned}
$$

Substitute (3.1.4.1) – (3.1.4.13) into (3.1.3),

$$(r-\frac{1}{2}v-\lambda m)ixf+\frac{1}{2}v(-x^2 f)+\kappa_v(\theta_v-v)B(\tau)f+\frac{1}{2}\sigma^2 vB^2(\tau)f+\rho\sigma vixB(\tau)f$$

$$+\kappa_\lambda(\theta_\lambda-\lambda)C(\tau)f+\frac{1}{2}\varepsilon^2\lambda C^2(\tau)f+\lambda\int_{-\infty}^{\infty}[(e^{ixy}-1)f]\phi_Y(y)dy-(ixr+A_\tau+B_\tau v+C_\tau\lambda)f=0$$

Let $ix=u,$ then

$$(r-\frac{1}{2}v-\lambda m)uf+\frac{1}{2}v(-x^2 f)+\kappa_v(\theta_v-v)B(\tau)f+\frac{1}{2}\sigma^2 vB^2(\tau)f+\rho\sigma vuB(\tau)f$$

$$+\kappa_\lambda(\theta_\lambda-\lambda)C(\tau)f+\frac{1}{2}\varepsilon^2\lambda C^2(\tau)f+\lambda\int_{-\infty}^{\infty}[(e^{uy}-1)f]\phi_Y(y)dy-(ur+A_\tau+B_\tau v+C_\tau\lambda)f=0.$$

Since $ix=u,$ then $(ix)^2=u^2$

$$i^2x^2=u^2$$

$$-x^2=u^2.$$

We have

$$\left[(r-\frac{1}{2}v-\lambda m)u+\frac{1}{2}vu^2+\kappa_v(\theta_v-v)B(\tau)+\frac{1}{2}\sigma^2 vB^2(\tau)+\rho\sigma vuB(\tau)+\kappa_\lambda(\theta_\lambda-\lambda)C(\tau)\right.$$

$$\left.+\frac{1}{2}\varepsilon^2\lambda C^2(\tau)+\lambda\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy-ru-A_\tau-B_\tau v-C_\tau\lambda\right]f=0$$

So,

$$(r-\frac{1}{2}v-\lambda m)u+\frac{1}{2}vu^2+\kappa_v(\theta_v-v)B(\tau)+\frac{1}{2}\sigma^2 vB^2(\tau)+\rho\sigma vuB(\tau)+\kappa_\lambda(\theta_\lambda-\lambda)C(\tau)$$

$$+\frac{1}{2}\varepsilon^2\lambda C^2(\tau)+\lambda\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy-ru-A_\tau-B_\tau v-C_\tau\lambda=0.$$

Then

$$ru-\frac{1}{2}vu-\lambda mu+\frac{1}{2}vu^2+\kappa_v\theta_v B(\tau)-\kappa_v B(\tau)v+\frac{1}{2}\sigma^2 vB^2(\tau)+\rho\sigma vuB(\tau)$$

$$+\kappa_\lambda\theta_\lambda C(\tau)-\lambda\kappa_\lambda C(\tau)+\frac{1}{2}\varepsilon^2\lambda C^2(\tau)+\lambda\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy-ru-A_\tau-B_\tau v-C_\tau\lambda=0.$$

That is

$$-\frac{1}{2}vu-\lambda mu+\frac{1}{2}vu^2+\kappa_v\theta_v B(\tau)-\kappa_v B(\tau)v+\frac{1}{2}\sigma^2 vB^2(\tau)$$
$$+\rho\sigma vuB(\tau)+\kappa_\lambda\theta_\lambda C(\tau)-\lambda\kappa_\lambda C(\tau)$$
$$+\frac{1}{2}\varepsilon^2\lambda C^2(\tau)+\lambda\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy=A_\tau+B_\tau v+C_\tau\lambda\,.$$

Consider

$$(-\frac{1}{2}vu+\frac{1}{2}vu^2-\kappa_v B(\tau)v+\frac{1}{2}\sigma^2 vB^2(\tau)+\rho\sigma vuB(\tau))$$
$$+\left(\frac{1}{2}\varepsilon^2\lambda C^2(\tau)+\lambda\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy-\lambda mu-\lambda\kappa_\lambda C(\tau)\right)$$
$$+\left(\kappa_v\theta_v B(\tau)+\kappa_\lambda\theta_\lambda C(\tau)\right)=A_\tau+B_\tau v+C_\tau\lambda.$$

We obtain

$$\left(-\frac{1}{2}u+\frac{1}{2}u^2-\kappa_v B(\tau)+\frac{1}{2}\sigma^2 B^2(\tau)+\rho\sigma uB(\tau)\right)v$$
$$+\left(\frac{1}{2}\varepsilon^2 C^2(\tau)+\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy-mu-\kappa_\lambda C(\tau)\right)\lambda$$
$$+\left(\kappa_v\theta_v B(\tau)+\kappa_\lambda\theta_\lambda C(\tau)\right)=A_\tau+B_\tau v+C_\tau\lambda.$$

This leads to the following system:

$$A_\tau=\kappa_v\theta_v B(\tau)+\kappa_\lambda\theta_\lambda C(\tau) \tag{3.1.5}$$

$$B_\tau=-\frac{1}{2}(u-u^2)-(\kappa_v-\rho\sigma u)B(\tau)+\frac{1}{2}\sigma^2 B^2(\tau) \tag{3.1.6}$$

$$C_\tau=\frac{1}{2}\varepsilon^2 C^2(\tau)-\kappa_\lambda C(\tau)-mu+\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy\,. \tag{3.1.7}$$

In the equation (3.1.6) become a Ricatti equation.

$$B_\tau=-\frac{1}{2}(u-u^2)-(\kappa_v-\rho\sigma u)B(\tau)+\frac{1}{2}\sigma^2 B^2(\tau).$$

Let $\quad B(\tau)=-\dfrac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}\,.$

We obtain

$$\frac{d}{d(\tau)}B(\tau)=-\frac{1}{2}(u-u^2)-(\kappa_v-\rho\sigma u)B(\tau)+\frac{1}{2}\sigma^2B^2(\tau)$$

or

$$\frac{d\left(-\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}\right)}{d\tau}=-\frac{1}{2}(u-u^2)-(\kappa_v-\rho\sigma u)\left(-\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}\right)+\frac{1}{2}\sigma^2\left(-\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}\right)^2 \qquad (3.1.6.1)$$

We have

$$\frac{-\frac{dG'(\tau)}{\frac{\sigma^2}{2}G(\tau)}}{d\tau}=-\left(\frac{\sigma^2}{2}G(\tau)\frac{d}{d\tau}(G'(\tau)-G'(\tau)\frac{d}{d\tau}(\frac{\sigma^2}{2}G(\tau))\right)\frac{1}{\left(\frac{\sigma^2}{2}G(\tau)\right)^2}$$

$$=-\left(\frac{\sigma^2}{2}G(\tau)G''(\tau)-G'(\tau)\frac{\sigma^2}{2}\frac{dG(\tau)}{d(\tau)}\right)\frac{1}{\left(\frac{\sigma^2}{2}G(\tau)\right)^2}$$

$$=-\left(\frac{\sigma^2}{2}G(\tau)G''(\tau)-\frac{\sigma^2}{2}(G'(\tau))^2\right)\frac{1}{\left(\frac{\sigma^2}{2}G(\tau)\right)^2}$$

$$=-\left(\frac{\sigma^2}{2}G(\tau)G''(\tau)-\frac{\sigma^2}{2}(G'(\tau))^2\right)\frac{1}{\frac{\sigma^4}{4}G^2(\tau)} \qquad (3.1.6.2)$$

Substitute (3.1.6.2) into (3.1.6.1)

$$-\left(\frac{\sigma^2}{2}G(\tau)G''(\tau)-\frac{\sigma^2}{2}(G'(\tau))^2\right)\frac{1}{\frac{\sigma^4}{4}G^2(\tau)}$$

$$=-\frac{1}{2}(u-u^2)-(\kappa_v-\rho\sigma u)\left(-\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}\right)+\frac{1}{2}\sigma^2\left(-\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}\right)^2.$$

We get

$$\frac{\sigma^2}{2}G(\tau)G''(\tau)\left(-\frac{1}{\frac{\sigma^4}{4}G^2(\tau)}\right)-\frac{\sigma^2}{2}(G'(\tau))^2\left(-\frac{1}{\frac{\sigma^4}{4}G^2(\tau)}\right)$$

$$=-\frac{1}{2}(u-u^2)+(\kappa_v-\rho\sigma u)\left(\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}\right)+\frac{1}{2}\sigma^2\left(\frac{(G'(\tau))^2}{\frac{\sigma^4}{4}G^2(\tau)}\right).$$

This is

$$\frac{\sigma^2}{2}G(\tau)G''(\tau)\left(-\frac{1}{\frac{\sigma^4}{4}G^2(\tau)}\right)-\frac{\sigma^2}{2}(G'(\tau))^2\left(-\frac{1}{\frac{\sigma^4}{4}G^2(\tau)}\right)+\frac{1}{2}(u-u^2)$$

$$-(\kappa_v-\rho\sigma u)\left(\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}\right)-\frac{1}{2}\sigma^2\left(\frac{(G'(\tau))^2}{\frac{\sigma^4}{4}G^2(\tau)}\right)=0$$

or

$$-\frac{\sigma^2}{2}\frac{G(\tau)G''(\tau)}{\frac{\sigma^4}{4}G^2(\tau)}-\frac{1}{2}(u^2-u)-(\kappa_v-\rho\sigma u)\left(\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}\right)=0$$

Multiply above equation by $-\frac{\sigma^2}{2}G(\tau)$,

$$-\frac{\sigma^2}{2}\frac{G(\tau)G''(\tau)}{\frac{\sigma^4}{4}G^2(\tau)}\left(-\frac{\sigma^2}{2}G(\tau)\right)-\frac{1}{2}(u^2-u)\left(-\frac{\sigma^2}{2}G(\tau)\right)-(\kappa_v-\rho\sigma u)\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}\left(-\frac{\sigma^2}{2}G(\tau)\right)=0$$

$$\frac{\sigma^4}{4}\frac{G^2(\tau)G''(\tau)}{\frac{\sigma^4}{4}G^2(\tau)}-\frac{1}{2}(u^2-u)\left(-\frac{\sigma^2}{2}G(\tau)\right)-(\kappa_v-\rho\sigma u)\left(-G'(\tau)\right)=0$$

$$G''(\tau)+\left(\frac{\sigma^2}{4}(u^2-u)G(\tau)\right)-(\kappa_v-\rho\sigma u)\left(-G'(\tau)\right)=0$$

$$G''(\tau)+\frac{\sigma^2}{4}(u^2-u)G(\tau)+(\kappa_v-\rho\sigma u)G'(\tau)=0$$

$$G''(\tau)+\left((\kappa_v-\rho\sigma u)G'(\tau)\right)+\frac{\sigma^2}{4}(u^2-u)G(\tau)=0.$$

Write down the characteristic equation

$$p=\frac{-(\kappa_V-\rho\sigma u)\pm\sqrt{(\kappa_V-\rho\sigma u)^2-4\left(\frac{\sigma^2}{4}(u^2-u)\right)}}{2}$$

$$p_1=\frac{-(\kappa_v-\rho\sigma u)-\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}}{2}$$

$$p_2=\frac{-(\kappa_v-\rho\sigma u)+\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}}{2}.$$

Next,

$$G(\tau)=C_1e^{p_1\tau}+C_2e^{p_2\tau}$$

$$=C_1e^{\frac{-(\kappa_v-\rho\sigma u)-\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}}{2}\tau}+C_2e^{\frac{-(\kappa_v-\rho\sigma u)+\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}}{2}\tau}$$

$$G(\tau)=C_1e^{-\frac{(\kappa_v-\rho\sigma u)+\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}}{2}\tau}+C_2e^{\frac{-(\kappa_v-\rho\sigma u)+\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}}{2}\tau}$$

$$=C_1e^{-\frac{1}{2}r_1\tau}+C_2e^{\frac{1}{2}r_2\tau}. \qquad (3.1.6.3)$$

where

$$r_1=(\kappa_v-\rho\sigma u)+H\ ,\ H=\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}$$

$$r_2=-(\kappa_v-\rho\sigma u)+H\ .$$

Note that $\qquad r_1+r_2=(\kappa_v-\rho\sigma u)+H-(\kappa_v-\rho\sigma u)+H=2H$

$$r_1r_2=\left[(\kappa_v-\rho\sigma u)+H\right]\left[-(\kappa_v-\rho\sigma u)+H\right]$$

$$=H^2-(\kappa_v-\rho\sigma u)^2$$

$$=(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)-(\kappa_v-\rho\sigma u)^2$$

$$=-\sigma^2(u^2-u).$$

The boundary condition

$$G(0)=C_1e^0+C_2e^0$$

$$=C_1(1)+C_2(1)$$

$$=C_1+C_2.$$

Equation (3.1.6.3) become

$$G'(\tau)=C_1\frac{de^{-\frac{1}{2}r_1\tau}}{d\tau}+C_2\frac{de^{\frac{1}{2}r_2\tau}}{d\tau}$$

$$=C_1e^{-\frac{1}{2}r_1\tau}\frac{d}{d\tau}(-\frac{1}{2}r_1\tau)+C_2e^{\frac{1}{2}r_2\tau}\frac{d}{d\tau}(\frac{1}{2}r_2\tau)$$

$$=C_1e^{-\frac{1}{2}r_1\tau}(-\frac{1}{2}r_1)\frac{d\tau}{d\tau}+C_2e^{\frac{1}{2}r_2\tau}\frac{1}{2}r_2\frac{d\tau}{d\tau}$$

$$=C_1e^{-\frac{1}{2}r_1\tau}(-\frac{1}{2}r_1)+C_2e^{\frac{1}{2}r_2\tau}\frac{1}{2}r_2.$$

From

$$G'(\tau) = -\frac{1}{2}C_1 e^{-\frac{1}{2}r_1\tau} r_1 + C_2 e^{\frac{1}{2}r_2\tau}\frac{1}{2}r_2,$$

we get

$$G'(0) = C_1 e^0(-\frac{1}{2}r_1) + C_2 e^0 \frac{1}{2}r_2$$

$$= C_1(-\frac{1}{2}r_1) + C_2\frac{1}{2}r_2$$

$$= -\frac{1}{2}r_1 C_1 + \frac{1}{2}r_2 C_2.$$

Let $G'(0) = 0$.

We have $-\frac{1}{2}r_1 C_1 + \frac{1}{2}r_2 C_2 = 0$

$$-r_1 C_1 + r_2 C_2 = 0$$

$$r_2 C_2 = r_1 C_1$$

$$\frac{r_2 C_2}{r_1} = C_1.$$

From $G(0) = C_1 + C_2$.

It imply

$$G(0) = \frac{r_2 C_2}{r_1} + C_2$$

and

$$C_2 = \frac{r_1 C_1}{r_2}$$

we obtain

$$G(0)r_1 = r_2 C_2 + C_2 r_1$$

$$= C_2(r_1 + r_2)$$

$$= C_2(2H).$$

That is

$$C_2 = \frac{r_1 G(0)}{2H}.$$

Note that

$$G(0) = C_1 + C_2$$

$$= C_1 + \frac{r_1 C_1}{r_2}.$$

Consider

$$r_2 G(0) = C_1 r_2 + \frac{(r_1 C_1) r_2}{r_2}$$

$$= C_1 r_2 + r_1 C_1$$

$$= C_1 (r_2 + r_1)$$

$$= C_1 (2H).$$

That is

$$C_1 = \frac{r_2 G(0)}{2H}.$$

We have

$$C_1 = \frac{r_2 G(0)}{2H} \quad \text{and} \quad C_2 = \frac{r_1 G(0)}{2H}.$$

Since

$$B(\tau) = -\frac{G'(\tau)}{\frac{\sigma^2}{2} G(\tau)},$$

then

$$B(\tau) = -\frac{C_1 e^{-\frac{1}{2} r_1 \tau} \left(-\frac{1}{2} r_1\right) + C_2 e^{\frac{1}{2} r_2 \tau} \frac{1}{2} r_2}{\frac{\sigma^2}{2} (C_1 e^{-\frac{1}{2} r_1 \tau} + C_2 e^{\frac{1}{2} r_2 \tau})}$$

$$B(\tau)=-\frac{\frac{r_2G(0)}{2H}e^{-\frac{1}{2}r_1\tau}\left(-\frac{1}{2}r_1\right)+\frac{r_1G(0)}{2H}e^{\frac{1}{2}r_2\tau}\,\frac{1}{2}r_2}{\frac{\sigma^2}{2}\left(\frac{r_2G(0)}{2H}e^{-\frac{1}{2}r_1\tau}+\frac{r_1G(0)}{2H}e^{\frac{1}{2}r_2\tau}\right)}$$

$$=\frac{-\frac{1}{2}r_1\frac{r_2G(0)}{2H}e^{-\frac{1}{2}r_1\tau}+\frac{1}{2}r_2\frac{r_1G(0)}{2H}e^{\frac{1}{2}r_2\tau}}{-\frac{\sigma^2}{2}\left[\frac{r_2G(0)}{2H}e^{-\frac{1}{2}r_1\tau}+\frac{r_1G(0)}{2H}e^{\frac{1}{2}r_2\tau}\right]}$$

$$=\frac{-\frac{G(0)}{2(2H)}(r_1r_2e^{-\frac{1}{2}r_1\tau}-r_1r_2e^{\frac{1}{2}r_2\tau})}{-\frac{\sigma^2G(0)}{2(2H)}(r_2e^{-\frac{1}{2}r_1\tau}+r_1e^{\frac{1}{2}r_2\tau})}$$

$$=\frac{1}{\sigma^2}\left(\frac{r_1r_2e^{-\frac{1}{2}r_1\tau}-r_2r_1e^{\frac{1}{2}r_2\tau}}{r_2e^{-\frac{1}{2}r_1\tau}+r_1e^{\frac{1}{2}r_2\tau}}\right)$$

$$=\frac{1}{\sigma^2}\left[\frac{-\sigma^2(u^2-u)e^{-\frac{1}{2}r_1\tau}+\sigma^2(u^2-u)e^{\frac{1}{2}r_2\tau}}{r_2e^{-\frac{1}{2}r_1\tau}+r_1e^{\frac{1}{2}r_2\tau}}\right]$$

$$=\frac{1}{\sigma^2}\left(-\sigma^2(u^2-u)\right)\left[\frac{e^{-\frac{1}{2}r_1\tau}-e^{\frac{1}{2}r_2\tau}}{r_2e^{-\frac{1}{2}r_1\tau}+r_1e^{\frac{1}{2}r_2\tau}}\right]$$

$$=-(u^2-u)\left(\frac{e^{-\frac{1}{2}r_2\tau}}{e^{-\frac{1}{2}r_2\tau}}\right)\left[\frac{e^{-\frac{1}{2}r_1\tau}-e^{\frac{1}{2}r_2\tau}}{r_2e^{-\frac{1}{2}r_1\tau}+r_1e^{\frac{1}{2}r_2\tau}}\right]$$

$$=-(u^2-u)\left[\frac{(e^{-\frac{1}{2}r_1\tau}e^{-\frac{1}{2}r_2\tau})-(e^{\frac{1}{2}r_2\tau}e^{-\frac{1}{2}r_2\tau})}{r_2(e^{-\frac{1}{2}r_1\tau}e^{-\frac{1}{2}r_2\tau})+r_1(e^{\frac{1}{2}r_2\tau}e^{-\frac{1}{2}r_2\tau})}\right]$$

$$=-(u^2-u)\left[\frac{e^{-\frac{(r_1+r_2)}{2}\tau}-e^0}{r_2e^{-\frac{(r_1+r_2)}{2}\tau}+r_1e^0}\right]$$

$$B(\tau) = -(u^2 - u)\left[\frac{e^{-\left(\frac{2H}{2}\right)\tau} - 1}{r_2 e^{-\left(\frac{2H}{2}\right)\tau} + r_1}\right]$$

$$= (u^2 - u)\left(\frac{1 - e^{-H\tau}}{r_1 + r_2 e^{-H\tau}}\right).$$

Next, consider equation (3.1.7).

$$C_\tau = \frac{1}{2}\varepsilon^2 C^2(\tau) - \kappa_\lambda C(\tau) - mu + \int_{-\infty}^{\infty} (e^{uy} - 1)\phi_Y(y)dy .$$

Let

$$C(\tau) = -\frac{M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)} .$$

Similarly in $B(\tau)$, we have

$$\frac{d}{d\tau}C(\tau) = \frac{1}{2}\varepsilon^2 C^2(\tau) - \kappa_\lambda C(\tau) - mu + \int_{-\infty}^{\infty} (e^{uy} - 1)\phi_Y(y)dy$$

$$\frac{d\left(-\frac{M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\right)}{d\tau} = \frac{1}{2}\varepsilon^2\left(-\frac{M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\right)^2 - \kappa_\lambda\left(-\frac{M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\right) - mu + \int_{-\infty}^{\infty} (e^{uy} - 1)\phi_Y(y)dy.$$

(3.1.7.1)

and

$$\frac{d\left(-\dfrac{M'(\tau)}{\dfrac{\varepsilon^2}{2}M(\tau)}\right)}{d\tau}=-\left(\frac{\varepsilon^2}{2}M(\tau)\frac{d}{d\tau}M'(\tau)-M'(\tau)\frac{d}{d\tau}(\frac{\varepsilon^2}{2}M(\tau))\right)\frac{1}{\left(\dfrac{\varepsilon^2}{2}M(\tau)\right)^2}$$

$$=-\left(\frac{\varepsilon^2}{2}M(\tau)M''(\tau)-M'(\tau)\frac{\varepsilon^2}{2}\frac{dM(\tau)}{d\tau}\right)\frac{1}{\left(\dfrac{\varepsilon^2}{2}M(\tau)\right)^2}$$

$$=-\left(\frac{\varepsilon^2}{2}M(\tau)M''(\tau)-\frac{\varepsilon^2}{2}(M'(\tau))^2\right)\frac{1}{\left(\dfrac{\varepsilon^2}{2}M(\tau)\right)^2}$$

$$=-\left(\frac{\varepsilon^2}{2}M(\tau)M''(\tau)-\frac{\varepsilon^2}{2}(M'(\tau))^2\right)\frac{1}{\dfrac{\varepsilon^4}{4}M^2(\tau)} \qquad (3.1.7.2)$$

Substitute (3.1.7.2) into (3.1.7.1)

$$-\left(\frac{\varepsilon^2}{2}M(\tau)M''(\tau)-\frac{\varepsilon^2}{2}(M'(\tau))^2\right)\frac{1}{\dfrac{\varepsilon^4}{4}M^2(\tau)}$$

$$=\frac{1}{2}\varepsilon^2\left(-\frac{M'(\tau)}{\dfrac{\varepsilon^2}{2}M(\tau)}\right)^2-\kappa_\lambda\left(-\frac{M'(\tau)}{\dfrac{\varepsilon^2}{2}M(\tau)}\right)-mu+\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy.$$

We have

$$\frac{\varepsilon^2}{2}M(\tau)M''(\tau)\left(-\frac{1}{\frac{\varepsilon^4}{4}M^2(\tau)}\right)-\frac{\varepsilon^2}{2}(M'(\tau))^2\left(-\frac{1}{\frac{\varepsilon^4}{4}M^2(\tau)}\right)$$

$$=\frac{1}{2}\varepsilon^2\left(\frac{-M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\right)^2-\kappa_\lambda\left(\frac{-M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\right)-mu+\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy.$$

That is

$$\frac{\varepsilon^2}{2}M(\tau)M''(\tau)\left(-\frac{1}{\frac{\varepsilon^4}{4}M^2(\tau)}\right)-\frac{\varepsilon^2}{2}(M'(\tau))^2\left(-\frac{1}{\frac{\varepsilon^4}{4}M^2(\tau)}\right)-\frac{1}{2}\varepsilon^2\left(\frac{-M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\right)^2$$

$$+\kappa_\lambda\left(\frac{-M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\right)+mu-\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy=0$$

or

$$-\frac{\varepsilon^2}{2}\left(\frac{M(\tau)M''(\tau)}{\frac{\varepsilon^4}{4}M^2(\tau)}\right)+\frac{\varepsilon^2}{2}\left(\frac{\left(M'(\tau)\right)^2}{\frac{\varepsilon^4}{4}M^2(\tau)}\right)-\frac{1}{2}\varepsilon^2\left(\frac{-M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\right)^2+\kappa_\lambda\left(\frac{-M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\right)$$

$$+mu-\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy=0.$$

We get

$$-\frac{\varepsilon^2}{2}\left(\frac{M(\tau)M''(\tau)}{\frac{\varepsilon^4}{4}M^2(\tau)}\right)+\kappa_\lambda\left(\frac{-M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\right)+mu-\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy=0.$$

Multiply above equation by $-\frac{\varepsilon^2}{2}M(\tau)$

$$-\frac{\varepsilon^2}{2}\left(\frac{M(\tau)M''(\tau)}{\frac{\varepsilon^4}{4}M^2(\tau)}\right)\left(-\frac{\varepsilon^2}{2}M(\tau)\right)+\kappa_\lambda\left(\frac{-M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\right)\left(-\frac{\varepsilon^2}{2}M(\tau)\right)$$

$$+mu\left(-\frac{\varepsilon^2}{2}M(\tau)\right)-\left(-\frac{\varepsilon^2}{2}M(\tau)\right)\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy=0.$$

Then

$$\frac{\varepsilon^4}{4}\left(\frac{M^2(\tau)M''(\tau)}{\frac{\varepsilon^4}{4}M^2(\tau)}\right)+\kappa_\lambda M'(\tau)+mu\left(-\frac{\varepsilon^2}{2}M(\tau)\right)+\left(\frac{\varepsilon^2}{2}M(\tau)\right)\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy=0$$

or

$$M''(\tau)+\kappa_\lambda M'(\tau)-mu\left(\frac{\varepsilon^2}{2}M(\tau)\right)+\left(\frac{\varepsilon^2}{2}M(\tau)\right)\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy=0$$

$$M''(\tau)+\kappa_\lambda M'(\tau)+M(\tau)\left(-mu\frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy\right)=0.$$

Write down the characteristic equation

$$r^2+de+e=0$$

$$r_1=\frac{-\kappa_\lambda-\sqrt{\kappa_\lambda^2-4(-mu\frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy)}}{2}$$

$$=\frac{-\kappa_\lambda-\sqrt{\kappa_\lambda^2-4\left(\frac{\varepsilon^2}{2}\right)\left(-mu+\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy\right)}}{2}$$

$$=\frac{-\kappa_\lambda-\sqrt{\kappa_\lambda^2-2\varepsilon^2\left(-mu+\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy\right)}}{2}$$

$$r_1=\frac{-\kappa_\lambda-\sqrt{\kappa_\lambda^2-2\varepsilon^2F}}{2}$$

where

$$F = -mu + \int_{-\infty}^{\infty} (e^{uy} - 1)\phi_Y(y)dy$$

So, $\quad r_1 = \dfrac{-\kappa_\lambda - \sqrt{\kappa_\lambda^2 - 2\varepsilon^2 F}}{2}$, $r_2 = \dfrac{-\kappa_\lambda + \sqrt{\kappa_\lambda^2 - 2\varepsilon^2 F}}{2}$.

Let $\quad E = \sqrt{\kappa_\lambda^2 - 2\varepsilon^2 F}$, $q_1 = -\kappa_\lambda + E$, $q_2 = \kappa_\lambda + E$.

We have

$$r_1 = -\frac{q_2}{2} \quad , \quad r_2 = \frac{q_1}{2}.$$

We obtain

$$\mathrm{M}(\tau) = C_1 e^{\frac{-q_2}{2}\tau} + C_2 e^{\frac{q_1}{2}\tau}.$$

Consider

$$\begin{aligned}
\mathrm{M}(0) &= C_1 e^0 + C_2 e^0 \\
&= C_1 + C_2.
\end{aligned}$$

Next,

$$\begin{aligned}
\mathrm{M}'(\tau) &= \frac{dC_1 e^{\frac{-q_2}{2}\tau}}{d\tau} + \frac{dC_2 e^{\frac{q_1}{2}\tau}}{d\tau} \\
&= C_1 \frac{de^{\frac{-q_2}{2}\tau}}{d\tau} + C_2 \frac{de^{\frac{q_1}{2}\tau}}{d\tau} \\
&= C_1 e^{\frac{-q_2}{2}\tau} \frac{d}{d\tau}\left(\frac{-q_2}{2}\tau\right) + C_2 e^{\frac{q_1}{2}\tau} \frac{d}{d\tau}\left(\frac{q_1}{2}\tau\right) \\
&= C_1 e^{\frac{-q_2}{2}\tau}\left(\frac{-q_2}{2}\right)\frac{d\tau}{d\tau} + C_2 e^{\frac{q_1}{2}\tau}\left(\frac{q_1}{2}\right)\frac{d\tau}{d\tau} \\
&= C_1 e^{\frac{-q_2}{2}\tau}\left(\frac{-q_2}{2}\right) + C_2 e^{\frac{q_1}{2}\tau}\frac{q_1}{2}.
\end{aligned}$$

Then

$$\mathrm{M}'(0) = C_1 e^0 \left(\frac{-q_2}{2}\right) + C_2 e^0 \ \frac{q_1}{2}$$

$$= C_1 \left(\frac{-q_2}{2}\right) + C_2 \ \frac{q_1}{2}$$

$$= \frac{-q_2}{2} C_1 + \frac{q_1}{2} C_2.$$

Let $M'(0) = 0$. We have $\frac{-q_2}{2} C_1 + \frac{q_1}{2} C_2 = 0$

Multiply above equation by 2, we get

$$\left(\frac{-q_2}{2} C_1\right) 2 + \left(\frac{q_1}{2} C_2\right) 2 = 0$$

$$-q_2 C_1 + q_1 C_2 = 0$$

$$q_1 C_2 = q_2 C_1$$

$$\frac{q_1 C_2}{q_2} = C_1.$$

From $\quad M(0) = C_1 + C_2$

$$= \frac{q_1 C_2}{q_2} + C_2$$

Then

$$q_2 M(0) = \frac{q_1 C_2}{q_2}(q_2) + q_2 C_2$$

$$= q_1 C_2 + q_2 C_2$$

$$= C_2 (q_1 + q_2).$$

Since

$$q_1 = \kappa_\lambda + E \quad \text{and} \quad q_2 = -\kappa_\lambda + E,$$

then

$$q_1 + q_2 = (\kappa_\lambda + E) + (-\kappa_\lambda + E)$$
$$= E + E$$
$$= 2E.$$

Thus

$$q_2 M(0) = C_2 2E$$
$$\frac{q_2 M(0)}{2E} = C_2.$$

We obtain

$$M(\tau) = C_1 e^{-\frac{q_2}{2}\tau} + C_2 e^{\frac{q_1}{2}\tau}$$
$$= \frac{q_1 C_2}{q_2} e^{-\frac{q_2}{2}\tau} + \frac{q_2 M(0)}{2E} e^{\frac{q_1}{2}\tau}$$
$$= \frac{q_1}{q_2}\left(\frac{q_2 M(0)}{2E}\right) e^{-\frac{q_2}{2}\tau} + \frac{q_2 M(0)}{2E} e^{\frac{q_1}{2}\tau}$$
$$= \frac{q_1 M(0)}{2E} e^{-\frac{q_2}{2}\tau} + \frac{q_2 M(0)}{2E} e^{\frac{q_1}{2}\tau}.$$

Thus

$$C(\tau) = -\frac{M'(\tau)}{\frac{\varepsilon^2}{2} M(\tau)}$$
$$= -\frac{C_1 e^{-\frac{q_2}{2}\tau}\left(\frac{-q_2}{2}\right) + C_2 e^{\frac{q_1}{2}\tau}\left(\frac{q_1}{2}\right)}{\frac{\varepsilon^2}{2}\left(C_1 e^{-\frac{q_2}{2}\tau} + C_2 e^{\frac{q_1}{2}\tau}\right)}$$
$$= -\frac{\frac{q_1}{q_2} C_2 e^{-\frac{q_2}{2}\tau}\left(\frac{-q_2}{2}\right) + \frac{q_2 M(0)}{2E} e^{\frac{q_1}{2}\tau}\left(\frac{q_1}{2}\right)}{\frac{\varepsilon^2}{2}\left(\frac{q_1}{q_2} C_2 e^{-\frac{q_2}{2}\tau} + \frac{q_2 M(0)}{2E} e^{\frac{q_1}{2}\tau}\right)}$$

$$C(\tau)=-\frac{\dfrac{q_1}{q_2}\left(\dfrac{q_2M(0)}{2E}\right)e^{-\frac{q_2}{2}\tau}\left(\dfrac{-q_2}{2}\right)+\dfrac{q_2M(0)}{2E}e^{\frac{q_1}{2}\tau}\left(\dfrac{q_1}{2}\right)}{\dfrac{\varepsilon^2}{2}\left(\dfrac{q_1}{q_2}\left(\dfrac{q_2M(0)}{2E}\right)e^{-\frac{q_2}{2}\tau}+\dfrac{q_2M(0)}{2E}e^{\frac{q_1}{2}\tau}\right)}$$

$$=-\frac{-\dfrac{1}{2}q_1\dfrac{q_2M(0)}{2E}e^{-\frac{q_2}{2}\tau}+\dfrac{1}{2}q_1\dfrac{q_2M(0)}{2E}e^{\frac{q_1}{2}\tau}}{\dfrac{\varepsilon^2}{2}\left[\dfrac{q_1M(0)}{2E}e^{-\frac{q_2}{2}\tau}+\dfrac{q_2M(0)}{2E}e^{\frac{q_1}{2}\tau}\right]}$$

$$=-\frac{\left(-\dfrac{1}{4}\right)\dfrac{M(0)}{E}\left(q_1q_2e^{\frac{-q_2}{2}\tau}-q_1q_2e^{\frac{q_1}{2}\tau}\right)}{\left(\dfrac{\varepsilon^2}{4}\right)\dfrac{M(0)}{E}\left(q_1e^{\frac{-q_2}{2}\tau}+q_2e^{\frac{q_1}{2}\tau}\right)}$$

$$=\frac{1}{\varepsilon^2}\left(\frac{q_1q_2e^{\frac{-q_2}{2}\tau}-q_1q_2e^{\frac{q_1}{2}\tau}}{q_1e^{\frac{-q_2}{2}\tau}+q_2e^{\frac{q_1}{2}\tau}}\right)$$

$$=\frac{1}{\varepsilon^2}\left(\frac{e^{-\frac{1}{2}q_1\tau}}{e^{-\frac{1}{2}q_1\tau}}\right)\left[\frac{q_1q_2e^{-\frac{1}{2}q_2\tau}-q_1q_2e^{\frac{1}{2}q_1\tau}}{q_1e^{-\frac{1}{2}q_2\tau}+q_2e^{\frac{1}{2}q_1\tau}}\right]$$

$$=\frac{1}{\varepsilon^2}\left[\frac{q_1q_2e^{-\frac{1}{2}q_2\tau}e^{-\frac{1}{2}q_1\tau}-q_1q_2e^{\frac{1}{2}q_1\tau}e^{-\frac{1}{2}q_1\tau}}{q_1e^{-\frac{1}{2}q_2\tau}e^{-\frac{1}{2}q_1\tau}+q_2e^{\frac{1}{2}q_1\tau}e^{-\frac{1}{2}q_1\tau}}\right]$$

$$=\frac{1}{\varepsilon^2}\left[\frac{q_1q_2e^{-\left(\frac{q_2+q_1}{2}\right)\tau}-q_1q_2e^{0}}{q_1e^{-\left(\frac{q_2+q_1}{2}\right)\tau}+q_1e^{0}}\right]$$

$$=\frac{1}{\varepsilon^2}\left[\frac{q_1q_2e^{-\left(\frac{2E}{2}\right)\tau}-q_1q_2}{q_1e^{-\left(\frac{2E}{2}\right)\tau}+q_2}\right]$$

$$=\frac{1}{\varepsilon^2}\left[\frac{q_1q_2e^{-E\tau}-q_1q_2}{q_1e^{-E\tau}+q_2}\right]$$

$$=\frac{1}{\varepsilon^2}q_1q_2\left[\frac{e^{-E\tau}-1}{q_1e^{-E\tau}+q_2}\right].$$

Next, we calculate $q_1 q_2$

$$q_1 q_2 = (\kappa_\lambda + E)(-\kappa_\lambda + E)$$

$$= \left[(\kappa_\lambda)(-\kappa_\lambda) + (E)(-\kappa_\lambda) + (\kappa_\lambda)(E) + (E)(E)\right]$$

$$= -\kappa_\lambda^2 - E\kappa_\lambda + \kappa_\lambda E + E^2$$

$$= -\kappa^2{}_\lambda + E^2$$

$$= -\kappa_\lambda^2 + \left(\sqrt{\kappa_\lambda^2 - 2\varepsilon^2 F}\right)^2$$

$$= -\kappa_\lambda^2 + \left(\kappa_\lambda^2 - 2\varepsilon^2 F\right).$$

That is

$$C(\tau) = \frac{1}{\varepsilon^2}\left(-2\varepsilon^2 F\right)\left[\frac{e^{-E\tau} - 1}{q_1 e^{-E\tau} + q_2}\right]$$

$$= -2F\left[\frac{e^{-E\tau} - 1}{q_1 e^{-E\tau} + q_2}\right]$$

$$= 2F\left[\frac{1 - e^{-E\tau}}{q_1 e^{-E\tau} + q_2}\right]$$

Consider equation (3.1.5)

$$A_\tau = \kappa_v \theta_v B(\tau) + \kappa_\lambda \theta_\lambda C(\tau)$$

Integrating with respect to τ

$$A(\tau) = \kappa_v \theta_v \int_0^t B(s)ds + \kappa_\lambda \theta_\lambda \int_0^t C(s)ds$$

$$= k_v \theta_v \int_0^t \left(-\frac{G'(s)}{\frac{\sigma^2}{2} G(s)}\right) ds + k_\lambda \theta_\lambda \int_0^t \left(-\frac{M'(s)}{\frac{\varepsilon^2}{2} M(s)}\right) ds$$

$$= -\frac{2\kappa_v \theta_v}{\sigma^2} \int_0^\tau \frac{G'(s)}{G(s)} ds - \frac{2\kappa_\lambda \theta_\lambda}{\varepsilon^2} \int_0^\tau \frac{M'(s)}{M(s)} ds$$

$$= -\frac{2\kappa_v \theta_v}{\sigma^2} \ln G(s)\Big|_{s=0}^{\tau} - \frac{2\kappa_v \theta_v}{\varepsilon^2} \ln M(s)\Big|_{s=0}^{\tau}$$

$$= -\frac{2\kappa_v \theta_v}{\sigma^2}\left[\ln G(\tau) - \ln G(0)\right] - \frac{2\kappa_\lambda \theta_\lambda}{\varepsilon^2}\left[\ln M(\tau) - \ln M(0)\right]$$

$$A(\tau) = -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\frac{G(\tau)}{G(0)} - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\frac{M(\tau)}{M(0)}$$

$$= -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\left[\frac{C_1 e^{-\frac{1}{2}r_1\tau} + C_2 e^{\frac{1}{2}r_2\tau}}{C_1 + C_2}\right] - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\left[\frac{C_1 e^{\frac{-q_2}{2}\tau} + C_2 e^{\frac{q_1}{2}\tau}}{C_1 + C_2}\right]$$

$$= -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\left[\frac{\frac{r_2 G(0)}{2H} e^{-\frac{1}{2}r_1\tau} + \frac{r_1 G(0)}{2H} e^{\frac{1}{2}r_2\tau}}{\frac{r_2 G(0)}{2H} + \frac{r_1 G(0)}{2H}}\right] - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\left[\frac{\frac{q_1 M(0)}{2E} e^{\frac{-q_2}{2}\tau} + \frac{q_2 M(0)}{2E} e^{\frac{q_1}{2}\tau}}{\frac{q_1 M(0)}{2E} + \frac{q_2 M(0)}{2E}}\right]$$

$$= -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\left[\frac{\frac{G(0)}{2H}}{\frac{G(0)}{2H}}\left(\frac{r_2 e^{-\frac{1}{2}r_1\tau} + r_1 e^{\frac{1}{2}r_2\tau}}{r_2 + r_1}\right)\right] - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\left[\frac{\frac{M(0)}{2E}}{\frac{M(0)}{2E}}\left(\frac{q_1 e^{\frac{-q_2}{2}\tau} + q_2 e^{\frac{q_1}{2}\tau}}{q_1 + q_2}\right)\right]$$

$$= -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\left[\frac{r_2 e^{-\frac{1}{2}r_1\tau} + r_1 e^{\frac{1}{2}r_2\tau}}{2H}\right] - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\left[\frac{q_1 e^{-\frac{1}{2}q_2\tau} + q_2 e^{\frac{1}{2}q_1\tau}}{2E}\right].$$

The proof is now completed.

3.2 A Formula for European Option Pricing

Let C denote the price at time t of European style call option on the current price of the underlying asset S_t with strike price K and expiration time T.

The terminal payoff of a European call option on the underlying stock price S_t with strike price K is

$$\max(S_T - K,\ 0).$$

This means that the holder will exercise his right only if $S_T > K$ and then his gain is $S_T - K$. Otherwise, if $S_T \le K$, then the holder will buy the underlying asset from the market and the value of the option is zero.

Assuming the risk-free interest rate r is constant over the lifetime of the option, the price of the European call at time t is equal to the discounted conditional expected payoff

$$C(t, S_T) = e^{-r(T-t)} E_M[\max(S_T - K, 0) | F_t].$$

Assume that $t=0$ and we define $L_T = \ln S_T$ and $k = \ln K$. Moreover, we express the call price option $C\left(0, \mathrm{S}_T\right)$ as a function of the log of the strike price K rather than the terminal log asset price S_T. The initial call value $C_T(k)$ is related to the risk-neutral density $q_T(l)$ by

$$C_T(k) = e^{-rT} \int_k^{\infty} (e^l - e^k) q_T(l) dl, \tag{3.2.1}$$

where $q_T(l)$ is the density function of the random variable L_T. It was mentioned by Carr and Madan [12] that $C_T(k)$ is not square integrable. To obtain a square integrable function, they introduced the modified call price function $c_T(k)$ defined by

$$c_T(k) = e^{\alpha k} C_T(k) \tag{3.2.2}$$

for some constant $\alpha > 0$ that makes $c_T(k)$ is square integrable in k over the entire real line and a good choice of α is that one fourth of the upper bound $E[S_T^{\alpha+1}] < \infty$. Consider the Fourier transform of $c_T(k)$

$$\begin{aligned}
\psi_T(u) &= \int_{-\infty}^{\infty} e^{iuk} c_T(k) dk \\
&= \int_{-\infty}^{\infty} e^{iuk} \int_k^{\infty} e^{\alpha k} e^{-rT} (e^l - e^k) q_T(l) dl dk \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(l) \int_{-\infty}^{l} (e^{l+\alpha k} - e^{(1+\alpha)k}) e^{iuk} dl dk \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(l) \left[\frac{e^{(\alpha+1+iu)l}}{\alpha + iu} - \frac{e^{(\alpha+1+iu)l}}{\alpha + iu + 1} \right] dl \\
&= e^{-rT} \int_{-\infty}^{\infty} \left[\frac{(\alpha+iu)e^{(\alpha+1+iu)l} + e^{(\alpha+1+iu)l} - (\alpha+iu)e^{(\alpha+1+iu)l}}{(\alpha+iu)(\alpha+iu+1)} \right] q_T(l) dl \\
&= e^{-rT} \int_{-\infty}^{\infty} \left[\frac{e^{(\alpha+1+iu)l}}{\alpha^2 + 2\alpha iu - u^2 + \alpha + iu} \right] q_T(l) dl \\
&= \frac{e^{-rT}}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} \int_{-\infty}^{\infty} e^{(\alpha+1+iu)l} q_T(l) dl \\
&= \frac{e^{-rT}}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} \int_{-\infty}^{\infty} e^{i(u-(\alpha+1)i)l} q_T(l) dl
\end{aligned}$$

$$\psi_T(u)=\frac{e^{-rT}f(l,v,\lambda,t;x=u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}$$

where f is the characteristic function defined in Theorem 3.1.

Lemma 3.2 Let $\alpha>0$. The Fourier transform of $c_T(k)$ exists if $E\left[S_T^{\alpha+1}\right]<\infty$.

Proof. Note that $E\left[S_T^{\alpha+1}\right]<\infty$ implies

$$\psi_T(0)<\infty, \tag{3.2.3}$$

since

$$\begin{aligned}\left|\psi_T(0)\right| &=\frac{e^{-rT}\left|f(-(\alpha+1)i)\right|}{\alpha^2+\alpha}\\ &=\frac{e^{-rT}E\left[S_T^{\alpha+1}\right]}{\alpha^2+\alpha},\end{aligned}$$

where the last equality follows from

$$\begin{aligned}\left|f(-(\alpha+1)i)\right| &=\left|E[e^{(-(\alpha+1)i)i\ln S_T}]\right|\\ &=\left|E[e^{(\alpha+1)\ln S_T}]\right|\\ &=E\left[S_T^{\alpha+1}\right].\end{aligned}$$

We have the equality

$$\psi_T(0)=\int_{-\infty}^{\infty}c_T(k)dk,$$

which follows from

$$\psi_T(u)=\int_{-\infty}^{\infty}e^{iuk}c_T(k)dk.$$

Combining this with (3.2.3) completes the proof.

Hence, the European call prices at time $t=0$ with strike price $k=\ln K$ can then be numerically obtained by using the inverse transform:

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\psi_T(u)du$$

$$= \frac{e^{-\alpha k}}{\pi}\int_{0}^{\infty} e^{-iuk}\,\frac{e^{-rT} f(l,v,\lambda,t;x=u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}du\ . \qquad (3.2.4)$$

Integration (3.2.4) is a direct Fourier transform and lends itself to an application of the Fast Fourier Transform (FFT), which has also done in Carr & Madan (1999).

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Appendix

Conference Proceeding

Paper Title “Jump-Diffusion with Stochastic Volatility and Intensity”

The 15th International Conference of International Academy Physical scicences .

December 9-13, 2012.

At Rajamangala University of Technology Thanyaburi, Thailand

By Rajamangala University of Technology Thanyaburi, Thailand

Jump-Diffusion with Stochastic Volatility and Intensity

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Abstract: An alternative option pricing model is proposed, in which the asset prices follow the jump-diffusion with stochastic volatility and intensity. The stochastic volatility follows the jump-diffusion. We find a formulation for the European-style option in terms of characteristic functions.

Keywords: Jump-diffusion model, Stochastic Volatility, Intensity, Characteristic functions.

1. Introduction

In 1973, Fischer Black and Myron Scholes introduced, a theoretical valuation formula for options is derived. In 1993, Heston studied a new technique to derive a closed – form solution for the price of a European call option on an asset with stochastic volatility. The Heston model assumes that S_t , the price of the asset, is determined by a stochastic process:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^S \tag{1}$$

where $\mu > 0$, v_t the instantaneous variance is a CIR process:

$$dv_t = \kappa_v(\theta_v - v_t)dt + \sigma\sqrt{v_t}dW_t^v \tag{2}$$

and $\kappa_v > 0, \theta_v > 0, \sigma > 0$, W_t^S, W_t^v are Brownian motion with correlation ρ .

In 1996, Bates introduced an efficient method is developed for pricing American options on stochastic volatility /jump-diffusion processes under systematic jump and volatility risk. The exchange rate S_t satisfy the following process:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^S + kdN_t \\ dv_t &= \kappa_v(\theta_v - v_t)dt + \sigma\sqrt{v_t}dW_t^v \end{aligned} \tag{3}$$

where k is the random percentage jump conditional on a jump occurring and N_t is a Poisson process with constant intensity λ .

2. Model Descriptions

The propose model assumes that the underlying asset has the following dynamics under risk-neutral measure,

$$\frac{dS_t}{S_t} = (r-\lambda_t m)dt + \sqrt{v_t}\,dW_t^S + Y_t dN_t$$
$$dv_t = \kappa_v(\theta_v - v_t)dt + \sigma\sqrt{v_t}\,dW_t^v$$
$$d\lambda_t = \kappa_\lambda(\theta_\lambda - \lambda_t)dt + \varepsilon\sqrt{v_t}\,dW_t^\lambda \tag{4}$$

where S_t, v_t, κ_v, θ_v, σ, Y_t, N_t, W_t^S and W_t^v are define (1), (2) and (3). r is the risk-free rate , m is the expected of Y_t, κ_λ is a mean-reverting rate. We assume that jump process N_t are independent of W_t^S, W_t^v and W_t^λ. A standard Brownian motion W_t^λ ,W_t^S and W_t^v are independent.

3. Characteristic Functions

Denote the characteristic function as

$$f(l,v,\lambda,t;x) = E[e^{ixX_T} \mid X_t = l, v_t = v] \tag{5}$$

where $T \geq t$ and $i=\sqrt{-1}$. Then, the following theorem holds.

Theorem 3.1 Suppose that S_t follows the dynamics in (4). Then the characteristic function for X_T defined in (5) is given by

$$f(l,v,\lambda,t;x) = \exp(ixl + ixr\tau + A(\tau) + B(\tau)v + C(\tau)\lambda),$$

where $A(\tau) = -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\left[\frac{r_2 e^{-\frac{1}{2}r_1\tau} + r_1 e^{\frac{1}{2}r_2\tau}}{2H}\right] - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\left[\frac{q_2 e^{-\frac{1}{2}q_1\tau} + q_1 e^{\frac{1}{2}q_2\tau}}{2E}\right]$,

$$B(\tau) = (u^2-u)\left(\frac{1-e^{-H\tau}}{r_1 + r_2 e^{-H\tau}}\right),\quad C(\tau) = 2F\left[\frac{1-e^{-E\tau}}{q_1 + q_2 e^{-E\tau}}\right],\quad u = ix$$

$$r_1 = (\kappa_v - \rho\sigma u) + H,\quad r_2 = -(\kappa_v - \rho\sigma u) + H,\quad H = \sqrt{(\kappa_v-\rho\sigma u)^2 - \sigma^2(u^2-u)}$$

$$q_1 = \kappa_\lambda + E,\quad q_2 = -\kappa_\lambda + E,\quad E = \sqrt{\kappa_\lambda^2 - 2\varepsilon^2 F},\quad F = -mu + \int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy$$

and $\phi_Y(y)$ is a density of random jump size Y_t.

Proof Feynman-Kac formula gives the following PDE for the characteristic function

$$(r - \frac{1}{2}v - \lambda m)f_l + \frac{1}{2}vf_{ll} + \kappa_v(\theta_v - v)f_v + \frac{1}{2}\sigma^2 v f_{vv} + \rho\sigma v f_{lv} + \kappa_\lambda(\theta_\lambda - v)f_\lambda$$
$$+\frac{1}{2}\varepsilon^2\lambda f_{\lambda\lambda} + \lambda\int_{-\infty}^{\infty}[f(l+y,v,\lambda,t;\phi) - f(l,v,\lambda,t;\phi)]\phi_Y(y)dy + f_t = 0, \tag{6}$$

$f(l,v,\lambda,T;x)=e^{ixl}$.

Consider form for the characteristic function:

$$f(l,v,\lambda,t;x)=\exp(ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda) \tag{7}$$

where $\tau=T-t$ and $A(\tau=0)=B(\tau=0)=C(\tau=0)$.

We plan to substitute equation (7) into equation (6). Firstly, we compute

$$f_l=ixf,\ f_{ll}=-x^2 f,\ f_v=B(\tau)f,\ f_{vv}=B^2(\tau)f,\ f_{lv}=ixB(\tau)f,\ f_\lambda=C(\tau)f,$$
$$f_{\lambda\lambda}=C^2(\tau)f,\ f_t=(-ixr-A_\tau-B_\tau v-C_\tau\lambda)f,$$
$$f(l+y,v,\lambda,t;x)-f(l,v,\lambda,t;x)=e^{ixy}f.$$

Substitute all terms above in equation (6),

$$(r-\frac{1}{2}v-\lambda m)ixf+\frac{1}{2}v(-x^2 f)+\kappa_v(\theta_v-v)B(\tau)f+\frac{1}{2}\sigma^2 vB^2(\tau)f+\rho\sigma vixB(\tau)f$$
$$+\kappa_\lambda(\theta_\lambda-\lambda)C(\tau)f+\frac{1}{2}\varepsilon^2\lambda C^2(\tau)f+\lambda f\int_{-\infty}^{\infty}e^{ixy}\phi_Y(y)dy-(ixr+A_\tau+B_\tau v+C_\tau\lambda)f=0.$$

Let $ix=u$, then

$$(r-\frac{1}{2}v-\lambda m)u+\frac{1}{2}vu^2+\kappa_v(\theta_v-v)B(\tau)+\frac{1}{2}\sigma^2 vB^2(\tau)+\rho\sigma vuB(\tau)$$
$$+\kappa_\lambda(\theta_\lambda-\lambda)C(\tau)+\frac{1}{2}\varepsilon^2\lambda C^2(\tau)+\lambda\int_{-\infty}^{\infty}e^{uy}\phi_Y(y)dy-ru-A_\tau-B_\tau v-C_\tau\lambda=0.$$

We have

$$\begin{aligned}A_\tau+B_\tau v+C_\tau\lambda=&\ \kappa_v\theta_v B(\tau)+\kappa_\lambda\theta_\lambda C(\tau)\\&+\left(\frac{1}{2}u^2-\frac{1}{2}u-\kappa_v B(\tau)+\frac{1}{2}\sigma^2 B^2(\tau)+\rho\sigma uB(\tau)\right)v\\&+\left(\frac{1}{2}\varepsilon^2 C^2(\tau)-\kappa_\lambda C(\tau)-mu+\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy\right)\lambda.\end{aligned}$$

This leads to the following system :

$$A_\tau=\kappa_v\theta_v B(\tau)+\kappa_\lambda\theta_\lambda C(\tau) \tag{8}$$

$$B_\tau=-\frac{1}{2}(u-u^2)-(\kappa_v-\rho\sigma u)B(\tau)+\frac{1}{2}\sigma^2 B^2(\tau) \tag{9}$$

$$C_\tau=\frac{1}{2}\varepsilon^2 C^2(\tau)-\kappa_\lambda C(\tau)-mu+\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy. \tag{10}$$

In the equation (9) become a Ricatti equation. Let

$$B(\tau)=-\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)},$$

substitute $B(\tau)$ in equation (9),

$$-\left[\frac{\sigma^2}{2}G(\tau)G''(\tau)-\frac{\sigma^2}{2}(G'(\tau))^2\right]\frac{1}{\frac{\sigma^4}{4}G^2(\tau)}=-\frac{1}{2}(u-u^2)+(\kappa_v-\rho\sigma u)\frac{G'(\tau)}{\frac{\sigma^2}{2}G^2(\tau)}+\frac{\frac{1}{2}\sigma^2(G'(\tau))^2}{\frac{\sigma^4}{4}G^2(\tau)}$$

Then

$$\frac{\sigma^2}{2}\frac{G(\tau)G''(\tau)}{\frac{\sigma^4}{4}G^2(\tau)}+\frac{1}{2}(u^2-u)-(\kappa_v-\rho\sigma u)\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}=0\,.$$

Multiply by $\frac{\sigma^2}{2}G(\tau)$,

$$G''(\tau)+(\kappa_v-\rho\sigma u)G'(\tau)+\frac{\sigma^2}{4}(u^2-u)G(\tau)=0\,.$$

General solution is

$$\begin{aligned}G(\tau)&=C_1e^{\frac{-(\kappa_v-\rho\sigma u)-\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}}{2}\tau}+C_2e^{\frac{-(\kappa_v-\rho\sigma u)+\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}}{2}\tau}\\&=C_1e^{\frac{-1}{2}r_1\tau}+C_2e^{\frac{1}{2}r_2\tau}\end{aligned}$$

where

$$r_1=(\kappa_v-\rho\sigma u)+H\ ,\ H=\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}$$
$$r_2=-(\kappa_v-\rho\sigma u)+H\,.$$

Note that $r_1+r_2=2H,\ r_1r_2=-\sigma^2(u^2-u)$.

The boundary condition

$$G(0)=C_1+C_2$$
$$G'(0)=\frac{-1}{2}r_1C_1+\frac{1}{2}r_2C_2=0\,.$$

We have $C_1=\frac{r_2G(0)}{2H}$ and $C_2=\frac{r_1G(0)}{2H}$.

Thus

$$\begin{aligned}B(\tau)=-\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}&=\frac{-\frac{1}{2}r_1\frac{r_2G(0)}{2H}e^{-\frac{1}{2}r_1\tau}+\frac{1}{2}r_2\frac{r_1G(0)}{2H}e^{\frac{1}{2}r_2\tau}}{-\frac{\sigma^2}{2}[\frac{r_2G(0)}{2H}e^{-\frac{1}{2}r_1\tau}+\frac{r_1G(0)}{2H}e^{\frac{1}{2}r_1\tau}]}\\&=\frac{1}{\sigma^2}\left[\frac{r_1r_2e^{-\frac{1}{2}r_1\tau}-r_1r_2e^{\frac{1}{2}r_2\tau}}{r_2e^{-\frac{1}{2}r_1\tau}+r_1e^{\frac{1}{2}r_2\tau}}\right]\\&=\frac{1}{\sigma^2}\left[\frac{-\sigma^2(u^2-u)e^{-\frac{1}{2}r_1\tau}+\sigma^2(u^2-u)e^{\frac{1}{2}r_2\tau}}{r_2e^{-\frac{1}{2}r_1\tau}+r_1e^{\frac{1}{2}r_2\tau}}\right]\end{aligned}$$

$$= (u^2 - u)\left[\frac{-e^{-\frac{1}{2}r_1\tau} + e^{\frac{1}{2}r_1\tau}}{r_2 e^{-\frac{1}{2}r_1\tau} + r_1 e^{\frac{1}{2}r_2\tau}}\right]$$

$$= (u^2 - u)\left(\frac{1 - e^{-H\tau}}{r_1 + r_2 e^{-H\tau}}\right).$$

Next, consider in equation (10).

$$C_\tau = \frac{1}{2}\varepsilon^2 C^2(\tau) - \kappa_\lambda C(\tau) - mu + \int_{-\infty}^{\infty}(e^{uy} - 1)\phi_Y(y)dy .$$

Let

$$C(\tau) = -\frac{M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)} .$$

Similarly in $B(\tau)$, we have

$$M(\tau) = \frac{q_2 M(0)}{2E}e^{-\frac{1}{2}q_1\tau} + \frac{q_1 M(0)}{2E}e^{\frac{1}{2}q_2\tau}$$

where $E = \sqrt{\kappa_\lambda^2 - 2\varepsilon^2 F}$, $F = -mu + \int_{-\infty}^{\infty}(e^{uy} - 1)\phi_Y(y)dy$, $q_1 = \kappa_\lambda + E$, $q_2 = -\kappa_\lambda + E$.

Thus

$$C(\tau) = \frac{-\frac{1}{2}q_1\frac{q_2 M(0)}{2E}e^{-\frac{1}{2}q_1\tau} + \frac{1}{2}q_2\frac{q_1 M(0)}{2E}e^{\frac{1}{2}q_2\tau}}{-\frac{\varepsilon^2}{2}[\frac{q_2 M(0)}{2E}e^{-\frac{1}{2}q_1\tau} + \frac{q_1 M(0)}{2E}e^{\frac{1}{2}q_2\tau}]}$$

$$= \frac{q_1 q_2 e^{-\frac{1}{2}q_1\tau} - q_1 q_2 e^{\frac{1}{2}q_2\tau}}{\varepsilon^2(q_2 e^{-\frac{1}{2}q_1\tau} + q_1 e^{\frac{1}{2}q_2\tau})}$$

$$= \frac{1}{\varepsilon^2}(2\varepsilon^2 F)\left[\frac{1 - e^{-E\tau}}{q_1 + q_2 e^{-E\tau}}\right]$$

$$= 2F\left[\frac{1 - e^{-E\tau}}{q_1 + q_2 e^{-E\tau}}\right].$$

Consider in equation (8),

$$A_\tau = \kappa_v\theta_v B(\tau) + \kappa_\lambda\theta_\lambda C(\tau) .$$

Integrating with respect to τ ,

$$A(\tau) = \kappa_v\theta_v\int_0^\tau B(s)ds + \kappa_\lambda\theta_\lambda\int_0^\tau C(s)ds$$

$$= -\frac{2\kappa_v\theta_v}{\sigma^2}\int_0^\tau\frac{G'(s)}{G(s)}ds - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\int_0^\tau\frac{M'(s)}{M(s)}ds$$

$$= -\frac{2\kappa_v\theta_v}{\sigma^2}\ln G(s)|_{s=0}^{\tau} - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln M(s)|_{s=0}^{\tau}$$
$$= -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\frac{G(\tau)}{G(0)} - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\frac{M(\tau)}{M(0)}$$

$$= -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\left[\frac{r_2 G(0)e^{-\frac{1}{2}r_1\tau}}{2HG(0)} + \frac{r_1 G(0)e^{\frac{1}{2}r_2\tau}}{2HG(0)}\right] - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\left[\frac{q_2 M(0)e^{-\frac{1}{2}q_1\tau}}{2EM(0)} + \frac{q_1 M(0)e^{\frac{1}{2}q_2\tau}}{2EM(0)}\right]$$

$$A(\tau) = -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\left[\frac{r_2 e^{-\frac{1}{2}r_1\tau} + r_1 e^{\frac{1}{2}r_2\tau}}{2H}\right] - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\left[\frac{q_2 e^{-\frac{1}{2}q_1\tau} + q_1 e^{\frac{1}{2}q_2\tau}}{2E}\right].$$

The proof is now completed.

4. A Formula for European Option Pricing

Following Carr and Madan (1999) , the modified call price $c_T(k)$ is defined by

$$c_T(k) = e^{\alpha k}C_T(k) \qquad \text{for some constant } \alpha > 0$$

where $C_T(k) = \int_k^\infty e^{-rT}(e^s - e^k)q_T(s)ds$ is the value of a T maturity call option with strike price e^k ($k = \ln K$), and $q_T(s)$ be the risk-neutral density of the log asset price $s_T = \ln S_T$. As $C_T(k)$ is not square integrable over $(-\infty,\infty)$, the introduction of a damping factor $e^{\alpha k}$ aims at removing this problem.

Theorems 3.2 The Fourier transform of $c_T(k)$ exist:

$$\psi_T(\xi) = \int_{-\infty}^{\infty} e^{i\xi k}c_T(k)dk$$

Proof

$$\psi_T(\xi) = \int_{-\infty}^{\infty} e^{i\xi k}\int_k^\infty e^{\alpha k}e^{-r_T}(e^s - e^k)q_T(s)dsdk$$
$$= \int_{-\infty}^{\infty} e^{-r_T}q_T(s)\int_{-\infty}^{s}(e^{\frac{(\alpha+1+i\xi)s}{\alpha+i\xi}} - e^{\frac{(\alpha+1+i\xi)s}{\alpha+1+i\xi}})ds$$

$$= \frac{e^{-rT}f(l,v,\lambda,t;x=\xi-(\alpha+1)i)}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}, \tag{11}$$

where f is the characteristic function defined in theorem 3.1

A sufficient condition for c_T to be square-intefrable is given by $\psi_T(0)$ being finite. This is equivalent to

$$E(S_T^{\alpha+1}) < \infty.$$

Call prices can then be numerically obtained by using the inverse transform:

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi k} \psi_T(\xi)\, d\xi$$

$$= \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-i\xi k} \psi_T(\xi)\, d\xi \qquad (12)$$

More precisely, the call price is determined by substituting (11) into (12) and performing the required integration. Integration (12) is a direct Fourier transform and lends itself to an application of the FFT.

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