ON SMALL PRINCIPALLY INJECTIVE RINGS

KITTISAK AMNUAYKARN

A THESIS SUBMITTED IN PARTIAL FULLFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE PROGRAM IN MATHEMATICS

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Thesis Title	On Small Principally Injective Rings
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Program	Mathematics
Thesis Advisor	Assistant Professor Sarun Wongwai, Ph.D.
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ABSTRACT

The purposes of this thesis are (1) to study properties and characterizations of small principally injective modules, (2) to study properties and characterizations of small principally injective rings, and (3) to find some relations between small principally injective modules, small principally injective rings and projective modules.

Let R be a ring. A right R-module M is called *principally injective* if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. A right R-module M is called *small principally injective* if every R-homomorphism from a small and principal right ideal of R to M can be extended to an R-homomorphism from R to M. R is called a right *small principally injective ring* if R_R is a small principally injective module.

The results were as follows. (1) Let R be a right small principally injective ring. Then (1.1) $lr(Ra) = Ra$ for any $a \in J(R)$. (1.2) If $aR \oplus bR$ and $Ra \oplus Rb$ are both direct, $a,b \in J(R)$, then $l(a)+l(b) = R$. (2) Let R be right small principally injective , $a \in R$ and $b \in J(R)$. (2.1) If bR embeds in aR, then Rb is an image of Ra. (2.2) If aR is an image of bR, then Ra embeds in Rb. (2.3) If $bR \cong aR$, then $Ra \cong Rb$. (3) The following conditions are equivalent for a ring R : (3.1) every small and principal right ideal of R is projective; (3.2) every factor module of a small principally injective module is small principally injective; (3.3) every factor module of an injective R-module is small principally injective. (4) Let R be right small principally injective and $b_i \in J(R)$, $(1 \le i \le n)$. (4.1) If $Rb_1 \oplus ... \oplus Rb_n$, is direct, then any $\alpha : Rb_1 + ... + Rb_n \to R$ can be extended to R. (4.2) If $b_1R \oplus ... \oplus b_nR$ is direct, then $R(b_1 + ... + b_n) = Rb_1 + ... + Rb_n$.

Keywords: principally injective rings, small principally injective rings

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Kittisak Amnuaykarn

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List of Abbreviations

$A \oplus B$	A direct sum B
$End_R(M)$	The set of R-homomorphism from M to M
F	Field F
$f : M \to N$	A function f from M to N
$f(M)$, $Im(f)$	Image of f
$Hom_R(M, N)$	The set of R-homomorphism from M to N
$Ker(f)$	Kernel of f
$J(M)$, $Rad(M_R)$	Jacobson radical of a right R-module M
$J(R) = Rad(R_R)$	Jacobson radical of a ring R
$J(S)$	Jacobson radical of a ring S
$J(S) \subset {}_S S_S$	$J(S)$ is an (two-side) ideal of ring S
$l_M(A)$	Left annihilator of A in M
M_R	M is a right R-module
$M_1 \times M_2$	Cartesian products of M_1 and M_2
M/K	A factor module of M modulo K or a factor module of M by K
$M \cong N$	M isomorphic N
R	Ring R
R_R	Ring R is a right R-module is called Regular right R-module
$r_R(X)$	Right annihilator of X in R
$Z(M)$	Singular submodule of M
1_M	Identity map on a module M
$\begin{pmatrix} F & F \\ F & F \end{pmatrix} = M_2(F)$	The set of all 2×2 matrices having elements of a field F as entries
$\eta : M \to M/K$	η (*eta*) is the natural epimorphism of M onto M/K
$\iota = \iota_{A \subset B} : A \to B$	ι (*iota*) is the inclusion map of A in B
π_j	π_j is the j-th projection map
$\forall$	For all
$\cap$	Intersection of set

List of Abbreviations (Continued)

$\not\subset$	is not subset
$\subset$	subset
$\in$	is in, member of set
$\subset^{e}$	Essential (Large)
$\ll$	Superfluous (Small)
$\prod_{i \in I} N_i$	Direct product of N_i
$\bigoplus_{i=1}^{n} N_i$	Direct sum of N_i

CHAPTER 1
INTRODUCTION

In modules and rings theory research field, there are three methods for doing the research. Firstly, to study about the fundamental of algebra and modules theory over arbitrary rings. Secondly, to study about the modules over special rings. Thirdly, to study about ring *R* by way of the categories of *R*-modules. Many mathematicians have concentrated on these methods.

1.1 Background and Statement of the Problems

Many generalizations of the injectivity were obtained, e.g., *principally injectivity* and *mininjectivity*. In [2], V. Camillo introduced the definition of principally injective modules by calling a right *R*-module *M* is *principally injective* if every *R*-homomorphism from a principal right ideal of *R* to *M* can be extended to an *R*-homomorphism from *R* to *M*. In [7], Nicholson and Yousif studied to the structure of principally injective rings and gave some applications of these rings. A ring *R* is called *right principally injective* if every *R*-homomorphism from a principal right ideal of *R* to *R* can be extended to an *R*-homomorphism from *R* to *R*. In [12], L.V. Thuyet, and T.C. Quynh introduced the definitions of a small principally module. A right *R*-module *M* is called *small principally injective* if every *R*-homomorphism from a small and principal right ideal *aR* to *M* can be extended to an *R*-homomorphism from *R* to *M*. In [10], N. V. Sanh, K. P. Shum, S. Dhompongsa and S. Wongwai introduced the definitions of quasi principally injective modules. A right *R*-module *M* is called *quasi-principally injective* if every *R*-homomorphism from an *M*-cyclic submodule of *M* to *M* can be extended to *M*.

1.2 Purpose of the Study

In this thesis, we have the purposes of study which are to extend concept of the previous works and to generalize new concepts which are :

1.2.1 To extend the concept of *principally injective rings*.

1.2.2 To generalize the concept of *small principally injective modules*.

1.2.3 To establish and extend some new concepts which are dual to *small principally-injective rings* and *small principally-injective modules*.

1.3 Research Questions and Hypothesis

We are interested in seeing to extend the characterizations and properties which remain valid from these previous concepts which can be extended from *principally injective rings*, *principally quasi-injective modules* [9], and *small -injective rings* [12]. In this research, we give characterizations and properties of these modules. A right R-module M is called *small principally injective* if every R-homomorphism from a small and principal right ideal aR to M can be extended to an R-homomorphism from R to M. If R_R is SP-injective modules, then we call R is SP-injective rings. In this research we give some properties and characterizations of SP-injective modules and SP-injective rings.

1.4 Theoretical Perspective

In this thesis, we use many of the fundamental theories which are concerned to the rings and modules research. By the concerned theories are :

1.4.1 The fundamental of algebra theories.

1.4.2 The basic properties of rings and modules theory.

1.5 Delimitations and Limitations of the Study

For this thesis, we have the scopes and the limitations of studying which are concerned to the previous works which are:

1.5.1 To study properties and and characterizations of SP-injective modules.

1.5.2 To study properties and and characterizations of SP-injective rings.

1.6 Significance of the Study

The advantage of education and studying in this research, we can improve and develop the concepts and knowledge in the algebra and modules research field.

CHAPTER 2
LITERATURE REVIEW

In this chapter we give notations, definitions and fundamental theories of the modules and rings theory which are used in this thesis.

2.1 Rings, Modules, Submodules and Endomorphism Rings

This section is assembled summary of various notations, terminology and some background theories which are concerned and used for this thesis.

2.1.1 Definition. [14] By a *ring* we mean a nonempty set R with two binary operations + and •, called *addition* and *multiplication* (also called *product*), respectively, such that

(1) $(R, +)$ is an additive abelian group.

(2) $(R, \bullet)$ is a multiplicative semigroup.

(3) Multiplication is distributive (on both sides) over addition; that is, for all $a, b, c \in R$, $a\bullet(b + c) = a\bullet b + a\bullet c$ and $(a + b)\bullet c = a\bullet c + b\bullet c$.

The two distributive laws are respectively called the *left distributive* law and the *right distributive* law.

A *commutative ring* is a ring R in which multiplication is commutative; i.e. if $a\bullet b = b\bullet a$ for all $a, b \in R$. If a ring is not commutative it is called *noncommutative*.

A *ring with unity* is a ring R in which the multiplicative semigroup $(R, \bullet)$ has an identity element; that is, there exists $e \in R$ such that $ea = a = ae$ for all $a \in R$. The element e is called *unity* or the *identity* element of R. Generally, the unity or identity element is denoted by 1. In this thesis, R will be an associative ring with identity.

2.1.2 Definition. [14] A nonempty subset I of a ring R is called an *ideal* of R if

(1) $a, b \in I$ implies $a - b \in I$.

(2) $a \in I$ and $r \in R$ imply $ar \in I$ and $ra \in I$.

2.1.3 Definition. [13] A subgroup I of $(R, +)$ is called a *left ideal* of R if $RI \subset I$, and a *right ideal* if $IR \subset I$.

2.1.4 Definition. [14] A right ideal I of a ring R is called *principal* if $I = aR$ for some $a \in R$.

2.1.5 Definition. [14] Let R be a ring, M an additive abelian group and $(m, r) \mapsto mr$, a mapping of $M \times R$ into M such that

(1) $mr \in M$

(2) $(m_1 + m_2)r = m_1r + m_2r$

(3) $m(r_1 + r_2) = mr_1 + mr_2$

(4) $(mr_1)r_2 = m(r_1r_2)$

(5) $m\bullet 1 = m$

for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. Then M is called a *right R-module*, often written as M_R. Often mr is called the *scalar multiplication* or just *multiplication* of m by r on right. We define left R-module similarly.

2.1.6 Definition. [13] Let M be a right R-module. A subgroup N of $(M, +)$ is called a *submodule* of M if N is closed under multiplication with elements in R, that is $nr \in N$ for all $n \in N$, $r \in R$. Then N is also a right R-module by the operations induced from $M : N \times R \to N$, $(n, r) \mapsto nr$, for all $n \in N$, $r \in R$.

2.1.7 Proposition. *A subset N of an R-module M is a submodule of M if and only if*

(1) $0 \in N$.

(2) $n_1, n_2 \in N$ *implies* $n_1 - n_2 \in N$.

(3) $n \in N$, $r \in R$ *implies* $nr \in N$.

Proof. See [15, Lemma 5.3]. □

2.1.8 Definition. [1] Let M be a right R-module and let K be a submodule of M. Then the set of cosets

$$M/K = \{ x + K \mid x \in M \}$$

is a right R-module relative to the addition and scalar multiplication defined via

$$(x + K) + (y + K) = (x + y) + K \quad \text{and} \quad (x + K)r = xr + K.$$

The additive identity and inverses are given by

$$K = 0 + K \quad \text{and} \quad -(x + K) = -x + K.$$

The module M/K is called (the *right R-factor module of*) *M modulo K* or the *factor module of M by K*.

2.1.9 Definition. [13] Let M and N be right R-modules. A function $f : M \to N$ is called an (*R-module*) *homomorphism* if for all $m, m_1, m_2 \in M$ and $r \in R$

$$f(m_1 r + m_2) = f(m_1)r + f(m_2).$$

Equivalently, $f(m_1 + m_2) = f(m_1) + f(m_2)$ and $f(mr) = f(m)r$.

The set of R-homomorphisms of M in N is denoted by $Hom_R(M, N)$. In particular, with this addition and the composition of mappings, $Hom_R(M, M) = End_R(M)$ becomes a ring, called the *endomorphism ring* of M and $f \in End_R(M)$ is called *an R-endomorphism.* [13, 6.4]

2.1.10 Definition. [1] Let $f : M \to N$ be an R-homomorphism. Then

(1) f is called *R-monomorphism* (or R-monic) if f is injective (one-to-one).

(2) f is called *R-epimorphism* (or R-epic) if f is surjective (onto).

(3) f is called *R-isomorphism* if f is bijective (one-to-one and onto).

Two modules M and N are said to be *R-isomorphic*, abbreviated $M \cong N$ in case there is an *R-isomorphism* $f : M \to N$.

2.1.11 Definition. [1] Let K be a submodule of M. Then the mapping $\eta_K : M \to M/K$ from M onto the factor module M/K defined by

$$\eta_K(x) = x + K \in M/K \qquad (x \in M)$$

is seen to be an R-epimorphism with kernel K. We call η_K the *natural epimorphism of M onto M/K.*

2.1.12 Definition. [1] Let $A \subset B$. Then the function $\iota = \iota_{A \subset B} : A \to B$ defined by $\iota = (1_{B|A}) : a \mapsto a$ for all $a \in A$ is called the *inclusion map* of A in B. Note that if $A \subset B$ and $A \subset C$, and if $B \neq C$, then $\iota_{A \subset B} \neq \iota_{A \subset C}$. Of course $1_A = \iota_{A \subset A}$.

2.1.13 Definition. [14] Let M and N be right R-modules and let $f : M \to N$ be an R-homomorphism. Then the set

$Ker(f) = \{ x \in M \mid f(x) = 0 \}$ is called the *kernel* of f

and

$f(M) = \{ f(x) \in N \mid x \in M \}$ is called the *homomorphic image* (or simply *image*) of M under f and is denoted by $Im(f)$.

2.1.14 Proposition. *Let M and N be right R-modules and let $f : M \to N$ be an R-homomorphism. Then*

(1) *$Ker(f)$ is a submodule of M.*

(2) *$Im(f) = f(M)$ is a submodule of N.*

Proof. See [13, 6.5]. □

2.1.15 Proposition. *Let M and N be right R-modules and let $f : M \to N$ be an R-isomorphism. Then the inverse mapping $f^{-1} : N \to M$ is an R-isomorphism.*

Proof. See[14, Chapter14, 3]. □

2.1.16 Theorem. *Let M, M', N and N' be right R-modules and let $f : M \to N$ be an R-homomorphism.*

(1) *If $g : M \to M'$ is an epimorphism with $Ker(g) \subset Ker(f)$, then there exists a unique homomorphism $h : M' \to N$ such that*

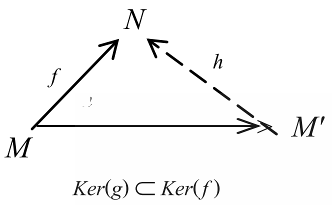
$$f = hg.$$

Moreover, $Ker(h) = g(Ker(f))$ and $Im(h) = Im(f)$, so that h is monic if and only if $Ker(g) = Ker(f)$ and h is epic if and only if f is epic.

(2) *If $g : N' \to N$ is a monomorphism with $Im(f) \subset Im(g)$, then there exists a unique homomorphism $h : M \to N'$ such that*

$$f = gh.$$

Moreover, $Ker(h) = Ker(f)$ *and* $Im(h) = g^{\leftarrow}(Im(f))$*, so that h is monic if and only if f is monic and h is epic if and only if* $Im(g) = Im(f)$.



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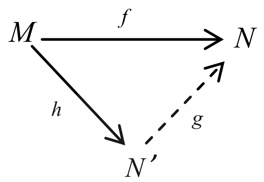


(2)

Proof. See [1, Chapter 1, 46]. □

2.1.17 Definition. [20] A submodule K of the module M is fully invariant in M if $f(K) \subset K$ for every endomorphism f of M.

2.2 Essential and Superfluous Submodules

In this section, we give the definitions of essential and superfluous submodules and some theories which are used in this thesis.

2.2.1 Definition. [13] A submodule K of M is called *essential* (or *large*) in M, abbreviated $K \subset^e M$, if for every submodule L of M, $K \cap L = 0$ implies $L = 0$.

2.2.2 Definition. [13] A submodule K of M is called *superfluous* (or *small*) in M, abbreviated $K \ll M$, if for every submodule L of M, $K + L = M$ implies $L = M$.

2.2.3 Proposition. *Let M be a right R-module with submodules* $K \subset N \subset M$ *and* $H \subset M$*. Then*

(1) $N \ll M$ *if and only if* $K \ll M$ *and* $N/K \ll M/K$;

(2) $H + K \ll M$ *if and only if* $H \ll M$ *and* $K \ll M$.

Proof. See [1, Proposition 5.17]. □

2.2.4 Proposition. *If $K \ll M$ and $f : M \to N$ is a homomorphism then $f(K) \ll N$. In particular, if $K \ll M \subset N$ then $K \ll N$.*

Proof. See [1, Proposition 5.18]. □

2.3 Annihilators and Singular Modules

In this section, we give the definitions of annihilators, singular modules and some theories which are used in this thesis.

2.3.1 Definition. [1] Let M be a right (resp. left) R-module. For each $X \subset M$, the *right* (resp. *left*) *annihilator* of X in R is defined by

$$r_R(X) = \{ r \in R \mid xr = 0, \forall x \in X \} \text{ (resp. } l_R(X) = \{ r \in R \mid rx = 0, \forall x \in X \}).$$

For a singleton $\{x\}$, we usually abbreviated to $r_R(x)$ (resp. $l_R(x)$).

2.3.2 Proposition. *Let M be a right R-module, let X and Y be subsets of M and let A and B be subsets of R. Then*

(1) *$r_R(X)$ is a right ideal of R.*

(2) *$X \subset Y$ imples $r_R(Y) \subset r_R(X)$.*

(3) *$A \subset B$ imples $l_M(B) \subset l_M(A)$.*

(4) *$X \subset l_M\, r_R(X)$ and $A \subset r_R\, l_M(A)$.*

Proof. See [1, Proposition 2.14 and Proposition 2.15]. □

2.3.3 Proposition. *Let M and N be right R-modules and let $f : M \to N$ be a homomorphism. If N' is an essential submodule of N, then $f^{-1}(N')$ is an essential submodule of M.*

Proof. See [4, Lemma 5.8(a)]. □

2.3.4 Proposition. *Let M be a right R-module over an arbitrary ring R, the set $Z(M) = \{ x \in M \mid r_R(x)$ is essential in $R_R \}$ is a submodule of M.*

Proof. See [4, Lemma 5.9]. □

2.3.5 Definition. [4] The submodule $Z(M) = \{ x \in M \mid r_R(x)$ is essential in $R_R \}$ is called the *singular submodule* of *M*. The module *M* is called a *singular module* if $Z(M) = M$. The module *M* is called *a nonsingular module* if $Z(M) = 0$.

2.4 Maximal and Minimal Submodules

In this section, we give the definitions and some properties of maximal submodules, minimal (or simple) submodules and some theories which are used in this thesis.

2.4.1 Definition. [13] A right *R*-module *M* is called *simple* if $M \neq 0$ and *M* has no submodules except 0 and *M*.

2.4.2 Definition. [13] A submodule *K* of *M* is called *maximal submodule* of *M* if $K \neq M$ and it is not properly contained in any proper submodules of *M*, i.e. *K* is *maximal in M* if, $K \neq M$ and for every $A \subset M$, $K \subset A$ implies $K = A$.

2.4.3 Definition. [13] A submodule *N* of *M* is called *minimal* (or *simple*) *submodule* of *M* if $N \neq 0$ and it has no non zero proper submodules of *M*, i.e. *N* is *minimal* (or *simple*) *in M* if $N \neq 0$ and for every nonzero submodules *A* of *M*, $A \subset N$ implies $A = N$.

2.4.4 Proposition. Let M and N be right R-modules. If $f : M \rightarrow N$ is an epimorphism with $\text{Ker}(f) = K$, then there is a unique isomorphism $\sigma : M/K \rightarrow N$ such that. $\sigma(m+K) = f(m)$ for all $m \in M$

Proof. See [1, Corollary 3.7]. □

2.4.5 Proposition. Let K be a submodule of M. A factor module M/K is simple if and only if K is a maximal submodule of M.

Proof. See [1, Corollary 2.10]. □

2.5 Injective and Projective Modules

In this section, we give the definitions of the injective modules, injective testing, projective modules and some theories which are used in this thesis.

2.5.1 Definition. [1] Let M be a right R-module. A right R-module U is called *injective relative to M* (or *U is M-injective*) if for every submodule K of M, for every homomorphism $\varphi : K \to U$ can be extended to a homomorphism $\alpha : M \to U$.

A right R-module U is said to be *injective* if it is M-injective for every right R-module M.

2.5.2 Proposition. *The following statements about a right R-module U are equivalent* :

(1) *U is injective*;

(2) *U is injective relative to R*;

(3) *For every right ideal $I \subset R_R$ and every homomorphism $h : I \to U$ there exists an $x \in U$ such that h is left multiplicative by x*

$$h(a) = xa \text{ for all } a \in I.$$

Proof. See [1, 18.3, Baer's Criterion]. □

2.5.3 Definition. [1] Let M be a right R-module. A right R-module U is called *projective relative to M* (or *U is M-projective*) if for every N_R, every epimorphism $g : M_R \to N_R$, for every homomorphism $\gamma : U_R \to N_R$ can be lifted to an R-homomorphism $\hat{\gamma} : U \to M$. A right R-module U is said to be *projective* if it is projective for every right R-module M.

2.5.4 Proposition. *Every right* (*resp. left*) *R-module can be embedded in an injective right* (*resp. left*) *R-module*.

Proof. See [1, Proposition 18.6]. □

2.6 Direct Summands and Product of Modules

Given two modules M_1 and M_2 we can construct their Cartesian product $M_1 \times M_2$. The structure of this product module is then determined "co-ordinatewise" from the factors $M_1 \times M_2$. For this section we give the definitions of direct summand, the projection and the injection maps, product of modules and some theories which are used in this thesis.

2.6.1 Definition. [1] Let M be a right R-module. A submodule X of M is called a *direct summand* of M if there is a submodule Y of M such that $X \cap Y = 0$ and $X + Y = M$. We write $M = X \oplus Y$; such that Y is also a *direct summand*.

2.6.2 Definition. [1] Let M_1 and M_2 be R-modules. Then with their products module $M_1 \times M_2$ are associated the natural injections and projections

$$\varphi_j : M_j \to M_1 \times M_2 \quad \text{and} \quad \pi_j : M_1 \times M_2 \to M_j$$

($j = 1, 2$), are defined by

$$\varphi_1(x_1) = (x_1, 0), \qquad \varphi_2(x_2) = (0, x_2)$$

and

$$\pi_1(x_1, x_2) = x_1, \qquad \pi_2(x_1, x_2) = x_2.$$

Moreover, we have

$$\pi_1\varphi_1 = 1_{M_1} \quad \text{and} \quad \pi_2\varphi_2 = 1_{M_2}$$

2.6.3 Definition. [1] Let A be a direct summand of M with complementary direct summand B, so $M = A \oplus B$. Then

$$\pi_A : a + b \mapsto a \ (a \in A, b \in B)$$

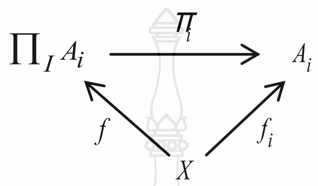
defines an epimorphism $\pi_A : M \to A$ is called *the projection of M on A along B.*

2.6.4 Definition. [13] Let $\{A_i , i \in I \}$ be a family of objects in the category C. An object P in C with morphisms $\{ \pi_i : P \to A_i \}$ is called the *product* of the family $\{A_i , i \in I \}$ if :

For every family of morphisms $\{ f_i : X \to A_i \}$ in the category C, there is a unique morphism $f : X \to P$ with $\pi_i f = f_i$ for all $i \in I$.

For the object P, we usually write $\prod_{i \in I} A_i$, $\prod_I A_i$ or $\prod A_i$. If all A_i are equal to A, then we put $\prod_I A_i = A^I$.

The morphism π_i are called the *i-projections* of the product. The definition can be described by the following commutative diagram :



2.6.5 Definition. [13] Let $\{ M_i , i \in I \}$ be a family of R-modules and ($\prod_{i \in I} M_i$, π_i) the product of the M_i . For $m, n \in \prod_{i \in I} M_i$, $r \in R$, using

$$\pi_i (m + n) = \pi_i (m) + \pi_i (n) \quad \text{and} \quad \pi_i(mr) = \pi_i (m)r,$$

a right R-module structure is defined on $\prod_{i \in I} M_i$ such that the π_i are homomorphisms. With this structure ($\prod_{i \in I} M_i$, π_i) is the product of the $\{ M_i , i \in I \}$ in R-module.

2.6.6 Proposition. *Properties*:

(1) *If* $\{ f_i : N \to M_i , i \in I \}$ *is a family of morphisms, then we get the map*

$$f : N \to \prod_{i \in I} M_i \text{ such that } n \mapsto (f_i(n))_{i \in I}$$

and $Ker(f) = \cap_I Ker(f_i)$ *since* $f (n) = 0$ *if and only if* $f_i(n) = 0$ *for all* $i \in I$.

(2) *For every* $j \in I$, *we have a canonical embedding*

$$\varepsilon_j : M_j \to \prod_{i \in I} M_i \text{, such that } m_j \mapsto (m_j\delta_{ji})_{i \in I} , m_j \in M_j ,$$

with $\varepsilon_j \pi_j = 1_{M_j}$, *i.e.* π_j *is a retraction and* ε_j *a coretraction.*

This construction can be extended to larger subsets of I : *For a subset A* $\subset I$ *we form the product* $\prod_{i\in A} M_i$ *and a family of homomorphisms*

$$f_j : \prod_{i\in A} M_i \to M_j, \qquad f_j = \begin{cases} \pi_j & for\ j \in A, \\ 0 & for\ j \in I - A. \end{cases}$$

Then there is a unique homomorphism

$$\varepsilon_A : \prod_{i\in A} M_i \to \prod_{i\in I} M_i \text{ with } \varepsilon_A \pi_j = \begin{cases} \pi_j & for\ j \in A, \\ 0 & for\ j \in I - A. \end{cases}$$

The universal property of $\prod_{i\in A} M_i$ *yields a homomorphism*

$$\pi_A : \prod_{i\in I} M_i \to \prod_{i\in A} M_i \text{ with } \pi_A\pi_j = \pi_j \text{ for } j \in I.$$

Together this implies $\varepsilon_A\pi_A\pi_j = \varepsilon_A\pi_j = \pi_j$ *for all* $j \in I$*, and by the properties of the product* $\prod_{i\in A} M_i$, *we get* $\varepsilon_A\pi_A = 1_{M_A}$.

Proof. See [13, 9.3, Properties (1), (2)] □

2.6.7 Definition. [1] We say $(M_\alpha)_{\alpha\in A}$ is independent in case for each $\alpha \in A$

$$M_\alpha \cap (\sum_{\beta\neq\alpha} M_\beta) = 0.$$

If the submodules $(M_\alpha)_{\alpha\in A}$ of M are independent, we say that the sum $\sum_A M_\alpha$ is *direct* and write

$$\sum_A M_\alpha = \bigoplus_A M_\alpha.$$

2.6.8 Proposition. [1] *Let* $(M_\alpha)_{\alpha\in A}$ *be an indexed set of submodules of a module M with inclusion maps* $(i_\alpha)_{\alpha\in A}$. *Then the following are equivalent*:

(a) $\sum_A M_\alpha$ *is the internal direct sum of* $(M_\alpha)_{\alpha\in A}$;

(b) $i = \bigoplus_A i_\alpha : \bigoplus_A M_\alpha \to M$ *is monic*;

(c) $(M_\alpha)_{\alpha\in A}$ *is independent*;

(d) $(M_\alpha)_{\alpha\in F}$ *is independent for every finite subset* $F \subset A$;

(e) *For every pair* $B, C \subset A$, *if* $B \cap C = \varnothing$, *then* $(\sum_B M_\beta) \cap (\sum_C M_\gamma) = 0$.

Proof. See [1, Proposition 6.10]. □

2.7 Generated and Cogenerated Classes

In this section, we give some definitions and theories of the generated and cogenerated classes which are concerned in this thesis.

2.7.1 Definition. [13] A subset *X* of a right *R*-module *M* is called a *generating set* of *M* if $XR = M$. We also say that *X generates M* or *M* is *generated by X*. If there is a finite generating set in *M*, then *M* is called *finitely generated*.

2.7.2 Definition. [1] Let U be a class of right *R*-modules. A module *M* is (*finitely*) *generated by* U (or U (*finitely*) *generates M*) if there exists an epimorphism

$$\bigoplus_{i\in I} U_i \to M$$

for some (finite) set *I* and $U_i \in$ U for every $i \in I$.

If U = $\{U\}$ is a singleton, then we say that *M* is (*finitely*) *generated by* U or (*finitely*) *U-generates*; this means that there exists an epimorphism

$$U^{(I)} \to M$$

for some (finite) set *I*.

2.7.3 Proposition. *If a module M has a generating set* $L \subset M$, *then there exists an epimorphism*

$$R^{(L)} \to M$$

Moreover, M is finitely R-generated if and only if M is finitely generated.

Proof. See [1, Theorem 8.1]. □

2.7.4 Definition. [17] Let *M* be a right *R*-module. A submodule *N* of *M* is said to be an *M-cyclic submodule* of *M* if it is the image of an endomorphism of *M*.

2.7.5 Definition. [1] Let U be a class of right R-modules. A module M is (*finitely*) *cogenerated by* U (or U (*finitely*) *cogenerates* M) if there exists a monomorphism

$$M \to \prod_{i \in I} U_i$$

for some (finite) set I and $U_i \in$ U for every $i \in I$.

If U = $\{U\}$ is a singleton, then we say that a module M is (*finitely*) *cogenerated by* U or (*finitely*) *U-cogenerates*; this means that there exists a monomorphism

$$M \to U^I$$

for some (finite) set I.

2.8 The Trace and Reject

In this section, we give some definitions and theories of the trace and reject which are concerned in this thesis.

2.8.1 Definition. [1] Let U be a class of right R-modules. The *trace* of U in M and the *reject* of U in M are defined by

$$Tr_M(\text{U}) = \sum\{ Im(h) \mid h : U \to M \text{ for some } U \in \text{U} \}$$

and

$$Rej_M(\text{U}) = \cap\{ Ker(h) \mid h : M \to U \text{ for some } U \in \text{U} \}.$$

If U = $\{U\}$ is a singleton, then the trace of U in M and the reject of U in M are in the form $Tr_M(U) = \sum\{ Im(h) \mid h \in Hom_R(U, M) \}$

and

$$Rej_M(U) = \cap\{ Ker(h) \mid h \in Hom_R(M, U) \}.$$

2.8.2 Proposition. *Let* U *be a class of right R-modules and let M be a right R-module.Then*

(1) Tr_M (U) *is the unique largest submodule L of M generated by* U;

(2) Rej_M(U) *is the unique smallest submodule K of M such that M*/*K is*

cogenerated by U.

Proof. See [1, Proposition 8.12]. □

2.9 Socle and Radical of Modules

In this section, we give some definitions and theories of the socle and radical of modules which are used in this thesis.

2.9.1 Definition. [13] Let M be a right R-module. The *socle* of M, $Soc(M)$, we denote the sum of all simple submodules of M. If there are no simple submodules in M we put $Soc(M) = 0$.

2.9.2 Definition. [13] Let M be a right R-module. The *radical* of M, $Rad(M)$, we denote the intersection of all maximal submodules of M. If M has no maximal submodules we set $Rad(M) = M$.

2.9.3 Proposition. *Let ε be the class of simple R-modules and let M be an R-module. Then*

$$\begin{aligned} Soc(M) &= Tr_M(\varepsilon) \\ &= \cap\{ L \subset M \mid L \text{ is essential in } M \}. \end{aligned}$$

Proof. See [13, 21.1]. □

2.9.4 Proposition. *Let ε be the class of simple R-modules and let M be an R-module. Then*

$$\begin{aligned} Rad(M) &= Rej_M(\varepsilon) \\ &= \sum\{ L \subset M \mid L \text{ is superfluous in } M \}. \end{aligned}$$

Proof. See [13, 21.5]. □

2.9.5 Proposition. *Let M be a right R-module. A right R-module M is finitely generated if and only if $Rad(M) \ll M$ and $M/Rad(M)$ is finitely generated.*

Proof. See [13, 21.6, (4)]. □

2.9.6 Proposition. *Let M be a right R-module. Then* $Soc(M) \subset^{e} M$ *if and only if every non-zero submodule of M contains a minimal submodule.*

Proof. See [1, Corollary 9.10]. □

2.10 The Radical of a Ring

In this section, we give some definitions and theories of the radical of a ring which are used in this thesis.

2.10.1 Definition. [1] Let R be a ring. The radical $Rad(R_R)$ of R_R is an (two side) ideal of R. This ideal of R is called the (*Jacobson*) *radical* of R, and we usually abbreviated by

$$J(R) = Rad(R_R).$$

Since $R = 1R$ is finite generated, $J(R) \ll R$. If $a \in J(R)$, then $aR \subset J(R) \ll R$ so $aR \ll R$. If $aR \ll R$, then $aR \subset J(R)$ and so $a \in aR \subset J(R)$. This shows that $a \in J(R)$ if and only if $aR \ll R$.

2.10.2 Definition. [1] Let R be a ring. An element $x \in R$ is called *right* (*left*) *quasi-regular* if $1 - x$ has a right (resp. left) inverse in R.

An element $x \in R$ is called *quasi-regular* if it is right and left quasi-regular.

A subset of R is said to be (*right*, *left*) *quasi-regular* if every element in it has the corresponding property.

2.10.3 Proposition. *Given a ring R for each of the following subsets of R is equal to the radical J(R) of R.*

(J_1) *The intersection of all maximal right* (*left*) *ideals of R*;

(J_2) *The intersection of all right* (*left*) *primitive ideals of R*;

(J_3) $\{ x \in R \mid$ *rxs is quasi-regular for all* $r, s \in R \}$;

(J_4) $\{ x \in R \mid$ *rx is quasi-regular for all* $r \in R \}$;

(J_5) $\{ x \in R \mid$ *xs is quasi-regular for all* $s \in R \}$;

(J_6) *The union of all the quasi-regular right* (*left*) *ideals of R*;

(J_7) *The union of all the quasi-regular ideals of* R;

(J_8) *The unique largest superfluous right (left) ideals of* R;

Moreover, (J_3), (J_4), (J_5), (J_6) *and* (J_7) *also describe the radical* $J(R)$ *if "quasi-regular" is replaced by "right quasi-regular" or by "left quasi-regular".*

Proof. See [1, Theorem 15.3]. □

2.10.4 Proposition. *Let* R *be a ring with radical* $J(R)$. *Then for every right* R-*module* M,

$$J(R)M_R \subset Rad(M_R).$$

If R *is semisimple modulo its radical, then for every right* R-*module,*

$$J(R)M_R = Rad(M_R)$$

and $M/J(R)M_R$ *is semisimple.*

Proof. See [1, Corollary 15.18]. □

CHAPTER 3
RESEARCH RESULT

In this chapter, we present the results of small principally injective modules and small principally injective injective rings.

3.1 *SP*-injective Modules

3.1.1 Definition. [12] Let R be a ring. A right R-module M is called *small principally injective* (briefly, *SP-injective*) if every R-homomorphism from a small and principal right ideal aR to M can be extended to an R-homomorphism from R to M.

3.1.2 Lemma. *Let M be right R-modules. Then M is SP-injective if and only if for each* $a \in J(R)$, $l_M r_R(a) = Ma$.

Proof. Clearly $Ma \subset l_M r_R(a)$. ($\Rightarrow$) Assume that M is SP-injective. Let $a \in J(R)$. To show that $l_M r_R(a) = Ma$. Let $x \in l_M r_R(a)$. Define $\varphi : aR \to xR$ by $\varphi(ar) = xr$, for every $r \in R$. To show that φ is the function. Let ar_1 and ar_2 be elements in aR such that $ar_1 = ar_2$. Then $ar_1 - ar_2 = 0$ and so $a(r_1 - r_2) = 0$. and $a(r_1 - r_2) = 0$, $x(r_1 - r_2) = 0$. Hence $xr_1 - xr_2 = 0$, then $xr_1 = xr_2$. Therefore $\varphi(ar_1) = xr_1 = xr_2 = \varphi(ar_2)$. This shows that φ *is well-defined*. Let $ar_1, ar_2 \in aR$ and $r \in R$. Then $\varphi(ar_1r + ar_2) = \varphi(a(r_1r + r_2)) = x(r_1r + r_2) = xr_1r + xr_2 = \varphi(ar_1)r + \varphi(ar_2)$. This shows that φ is an R-homomorphism. Since M is SP-injective, there exists an R-homomorphism $\hat{\varphi} : R \to M$ such that $\hat{\varphi} i_2 = i_1 \varphi$ where $i_1 : xR \to M$ and $i_2 : aR \to R$ are the inclusion maps. Then $x = \varphi(a) = \hat{\varphi}(a) = \hat{\varphi}(1.a) = \hat{\varphi}(1)a \in Ma$.

($\Leftarrow$) Let $a \in J(R)$, and let $\varphi : aR \to M$ be an R-homomorphism. Then $\varphi(a) \in l_M r_R(a)$, so by assumption, we have $\varphi(a) = xa$ for some $x \in M$. Define $\hat{\varphi} : R \to M$ by $\hat{\varphi}(r) = xr$ every $r \in R$. It is clear that $\hat{\varphi}$ is an R-homomorphism and is an extension of φ. □

3.1.3 Example. *Let* $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ *where F is a field,* $M_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. *Then M is SP-injective.*

Proof. We have only $A_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, $A_3 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$, $A_5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $A_6 = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ are right ideal of R, and we see that only $A_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is only the nonzero small principal right ideal of R because for every $A_i \subset R$, $2 \le i \le 5$, $A_i \ne R$ then $A_1 + A_i \ne R$. Since, for each $x = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = A_1$, $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, i.e., $xR = A_1$, A_1 is a principal right ideal of R. Let $\varphi: A_1 \to M$ be an R-homomorphism. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in A_1$, there exists $x_{11}, x_{12} \in F$ such that $\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}$. Then $\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}$. Then $\begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}$ so $x_{11} = 0$. Define $\hat{\varphi}: R \to M$ by $\hat{\varphi}\left(\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\right) = \begin{pmatrix} x_{12}a_{11} & x_{12}a_{12} \\ 0 & 0 \end{pmatrix}$ for every $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in R$. To show that $\hat{\varphi}$ is well-defined. Let $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} \in R$ such that $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}$. Then $\hat{\varphi}\left(\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\right) = \begin{pmatrix} x_{12}a_{11} & x_{12}a_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{12}b_{11} & x_{12}b_{12} \\ 0 & 0 \end{pmatrix} = \hat{\varphi}\left(\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}\right)$. To show that $\hat{\varphi}$ is an R-homomorphism. Let $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and $\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \in R$. Then $\hat{\varphi}\left(\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix} a_{11}r_1 + b_{11} & a_{11}r_2 + a_{12}r_3 + b_{12} \\ 0 & a_{22}r_3 + b_{22} \end{pmatrix}\right) = \begin{pmatrix} x_{12}(a_{11}r_1 + b_{11}) & x_{12}(a_{22}r_3 + b_{22}) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{12}a_{11}r_1 + x_{12}b_{11} & x_{12}a_{22}r_3 + x_{12}b_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{12}a_{11}r_1 & x_{12}a_{22}r_3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} x_{12}b_{11} & x_{12}b_{22} \\ 0 & 0 \end{pmatrix} = \hat{\varphi}\left(\begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix}\right) + \hat{\varphi}\left(\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}\right) + \hat{\varphi}\left(\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}\right) =$

$\hat{\varphi}\left(\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\right)\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} + \hat{\varphi}\left(\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}\right)$. To show that $\hat{\varphi}\,\iota = \varphi$. Let $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in A_1$. Then

$\hat{\varphi}\,\iota\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = \hat{\varphi}\left(\iota\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & x_{12}x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$

$= \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right)$. This shows that $\hat{\varphi}$ is an extension of φ. Thus M is *SP*-injective. □

3.1.4 Proposition. *Let M be $\{ M_i , i \in I \}$ be a family of right R-modules. Then the direct product $\prod_{i\in I} M_i$ is SP-injective if and only if each M_i is SP-injective.*

Proof. $(\Rightarrow)$ Let $\{ M_i , i \in I \}$ be a family of right R-modules and the direct product $\prod_{i\in I} M_i$ is *SP*-injective. Let $i \in I$, we must show that M_i is *SP*-injective.Let $a \in R$, $aR \ll R$ and let $\varphi : aR \to M_i$ be an R-homomorphism. Let π_i and φ_i, for each $i \in I$, be the i-th projection map and the i-th injection map, respectively. Since $\prod_{i\in I} M_i$ is *SP*-injective, there exists an R-homomorphism $\hat{\varphi} : R \to \prod_{i\in I} M_i$ such that $\hat{\varphi}\,_i = \varphi_i\varphi$ where $\iota : aR \to R$ is the inclusion map. Thus $\pi_i\hat{\varphi}\,\iota = \pi_i\varphi_i\varphi$, so by Definition 2.6.2, $\pi_i\hat{\varphi}\,\iota = \varphi$. Thus $\pi_i\hat{\varphi}$ is an extension of φ.

$(\Leftarrow)$ Let M_i is SP – injective. Let $a \in R$, $aR \ll R$ and let $\varphi : aR \to \prod_{i\in I} M_i$ be an R-homomorphism. Let π_i be the i-th projection map. Since, for each i, M_i is *SP*-injective, there exists an R-homomorphism $\alpha_i : R \to M_i$ such that $\pi_i\varphi = \alpha_i\iota$ where $\iota : aR \to R$ is the inclusion map. Then by Definition 2.6.5 and Proposition 2.6.6, we obtain $\hat{\varphi} : M \to \prod_{i\in I} N_i$ such that $\pi_i\hat{\varphi} = \alpha_i$ for each $i \in I$. Then $\pi_i\hat{\varphi}\,\iota = \alpha_i\,\iota$, so $\pi_i\varphi = \alpha_i\,\iota = \pi_i\hat{\varphi}\,\iota$. Hence $\pi_i\varphi = \pi_i\hat{\varphi}\,\iota$ for each $i \in I$. Therefore $\varphi = \hat{\varphi}\,\iota$. □

3.1.5 Lemma. *Let M_i $(1 \le i \le n)$ be SP-injective modules. Then $\bigoplus_{i=1}^{n} M_i$ is SP-injective if and only if each M_i is SP-injective.*

Proof. ($\Rightarrow$) Assume that M_i $(1 \le i \le n)$ be R-modules and $\overset{n}{\underset{i=1}{\oplus}} M_i$ is SP-injective. Let $i \in I$, we must show that M_i is SP-injective. Let $a \in R$, $aR \ll R$ and let $\varphi: aR \to M_i$ be an R-homomorphism. Let π_i and φ_i for each $i \in I$, be the i-th projection map and the i-th injection map, respectively. Since $\overset{n}{\underset{i=1}{\oplus}} M_i$ is SP-injective, there exists an R-homomorphism $\hat{\varphi}: R \to \overset{n}{\underset{i=1}{\oplus}} M_i$ such that $\hat{\varphi}\iota = \varphi_i\varphi$ where $\iota : aR \to R$ is the inclusion map. Thus $\pi_i\hat{\varphi}\iota = \pi_i\varphi_i\varphi$, so by Definition 2.6.2, $\pi_i\hat{\varphi}\iota = \varphi$. Thus $\pi_i\hat{\varphi}$ is an extension of φ.

($\Leftarrow$) Let $a \in J(R)$ and $\varphi : aR \to \overset{n}{\underset{i=1}{\oplus}} M_i$ be an R-homomorphism. Since for each $i \in \{1, 2, 3, \dots, n\}$, M_i is SP-injective, there exists an R-homomorphism $\varphi_i: R \to M_i$ such that $\varphi_i\ \iota = \pi_i\varphi$ where π_i is the i-th projection map from $\overset{n}{\underset{i=1}{\oplus}} M_i$ to M_i and $\iota : aR \to R$ is the inclusion map. Set $\hat{\varphi} = \iota_1\varphi_1 + \iota_2\varphi_2 + \dots + \iota_n\varphi_{i\,n} : R \to \overset{n}{\underset{i=1}{\oplus}} M_i$ where $\iota_i : M_i \to \overset{n}{\underset{i=1}{\oplus}} M_i$ for each $i \in \{1, 2, 3, \dots, n\}$ is the i-injection map. We must show that $\hat{\varphi}$ is an extension of φ. Let $a(r) \in s(R)$. Then $\hat{\varphi}\,\iota(a(r)) = \hat{\varphi}(a(r)) = \iota_1\varphi_1(a(r)) + \iota_2\varphi_2(a(r)) + \dots + \iota_n\varphi_n(a(r)) = \varphi_1(a(r)) + \varphi_2(a(r)) + \dots + \varphi_n(a(r)) = \varphi_1\iota_1(a(r)) + \varphi_2\iota_2(a(r)) + \dots + \varphi_n\iota_n(a(r)) = \pi_1\varphi(a(r)) + \pi_2\varphi(a(r)) + \dots + \pi_n\varphi(a(r)) = (\pi_1+\pi_2+\dots+\pi_n)\varphi(a(r)) = \varphi(a(r))$. Then $\overset{n}{\underset{i=1}{\oplus}} M_i$ is SP-injective. □

3.1.6 Lemma. *Any direct summand of SP-injective module is again SP-injective.*

Proof. Let M be an SP-injective module and let A be a direct summand of M. To show that A is an SP-injective. Let $a \in R$, $aR \ll R$ and let $\varphi: aR \to A$ be an R-homomorphism. Since M is SP-injective, there exists an R-homomorphism $\hat{\varphi}: R \to M$ such that $\alpha\varphi = \hat{\varphi}\iota$ where $\iota : aR \to R$ is the inclusion map and $\alpha : A \to M$ is the injection map. Let $\pi : M \to A$ be the projection map. Then $\pi\alpha\varphi = \pi\hat{\varphi}\iota$. Hence by Definition 2.6.2, $\varphi = \pi\hat{\varphi}\iota$. Then $\pi\hat{\varphi}$ is an extension of φ. □

3.2 *SP* - injective Rings

If R_R is an *SP*-injective modules, then we call R is a right *SP-injective* ring. In this section, we give some properties and characterizations of *SP*-injective rings.

3.2.1 Lemma. [12] *The following conditions are equivalent for a ring R*

(1) *R is right SP-injective ring.*

(2) $lr(a) = Ra$ *for any* $a \in J(R)$.

(3) $r(a) \subset r(b)$, *where* $a \in J(R)$, $b \in R$ *implies* $Rb \subset Ra$.

(4) $l(r(a) \cap bR) = l(b) + Ra$ *for all* $a \in J(R)$, $b \in R$.

(5) *If* $\alpha : aR \rightarrow R$, $a \in J(R)$, *is an R-homomorphism,* then $\alpha(a) \in Ra$.

3.2.2 Theorem. Let R be a right *SP-injective* ring. Then

(1) $lr(Ra) = Ra$, for any $a \in J(R)$.

(2) If $aR \oplus bR$ and $Ra \oplus Rb$ are both direct, $a,b \in J(R)$, then $l(a)+l(b) = R$.

Proof. (1) Let R be a right *SP*-injective ring and let $a \in J(R)$. To show that $lr(Ra) = Ra$.($\supset$) Let $ra \in Ra$. To show that $ra \in lr(Ra)$. Let $s \in R$, and $Ras = 0$. Then $ras = 0$ and hence $ra \in lr(Ra)$. ($\subset$) Let $x \in lr(Ra)$. Define $\varphi : aR \rightarrow xR$ by $\varphi(ar) = xr$, for every $r \in R$. To show that φ is the function. Let $ar = 0$ then $\varphi(ar) = xr = 0$. This shows that φ is well–defined. Let ar_1, $ar_2 \in aR$ and $r \in R$. Then $\varphi(ar_1r + ar_2) = \varphi(a(r_1r + r_2)) = x(r_1r + r_2) = xr_1r + xr_2 = \varphi(ar_1)r + \varphi(ar_2)$. This shows that φ is an R-homomorphism. Since R is a right *SP*-injective ring. Then there exists $\hat{\varphi} : R \rightarrow R$ an R-homomorphism, such that $i_1\varphi = \hat{\varphi}\, i_2$ where $i_1 : xR \rightarrow R$ and $i_2 : aR \rightarrow R$ are the inclusion maps. Then $x = \varphi(a) = \hat{\varphi}(a) = \hat{\varphi}(1.a) = \hat{\varphi}(1)a \in Ra$.

(2) Let R be a right *SP*-injective ring, a , $b \in J(R)$ and let $aR \oplus bR$ and $Ra \oplus Rb$ are both direct. To show that $l(a)+l(b) = R$. Define $\varphi : (a+b)R \rightarrow R$ by $\varphi(a+b)r = br$, for every $r \in R$. To show that φ is the function. If $(a+b) = 0$, then $ar = br \in aR \cap bR = 0$ so $br = 0$. Then $\varphi(a+b)r = br = 0$. This shows that φ is well–defined.

We now show that φ is an R-homomorphism. Let $(a+b)r_1$, $(a+b)r_2 \in (a+b)R$ and $r \in R$. Then $\varphi((a+b)r_1r + (a+b)r_2) = \varphi((a+b)(r_1r + r_2)) = b(r_1r + r_2) = br_1r + br_2 = \varphi(a+b)r_1\text{r} + \varphi(a+b)r_2$. This shows that φ is an R-homomorphism. Since R is a right SP-injective, there exists an R-homomorphism $\hat{\varphi} : R \to R$ such that $\varphi = \hat{\varphi}\, i$ where $i : (a+b)R \to R$ is the inclusion map. Hence $\hat{\varphi}(1)(a+b) = \hat{\varphi}(1.(a+b)) = \hat{\varphi}(a+b) = \varphi(a+b) = b$ so $\hat{\varphi}(1)(a+b) = b$. Then $\hat{\varphi}(1)a + \hat{\varphi}(1)b = b$, and so $\hat{\varphi}(1)a = b - \hat{\varphi}(1)b = (1-\hat{\varphi}(1))b \in Ra \cap Rb = 0$. Then $\hat{\varphi}(1) \in l(a)$ and $(1-\hat{\varphi}(1)) \in l(b)$. Hence $1 = \hat{\varphi}(1) + (1-\hat{\varphi}(1)) \in l(a) + l(b)$. Then $1 \in l(a) + l(b)$ so $l(a) + l(b) = R$. □

3.2.3 Proposition. If R is a right *SP-injective*, so is eRe for all $e^2 = e \in R$ satisfying $ReR = R$.

Proof. Let R be a right SP-injective and e be an idempotent sastisfying $ReR = R$. Write $S = eRe$. Let $a \in J(eSe)$ and let $\varphi : aS \to S$ be an S-homomorphism. To show that $r(a) \subset r(\varphi(a))$. Let $x \in r(a)$. Then $ax = 0$. Hence $\varphi(a)x = \varphi(ax) = \varphi(0) = 0$. This shows that $r(a) \subset r(\varphi(a))$, so $lr\varphi(a) \subset lr(a)$ by proposition 2.3.2 (3). Since $a(eRe)R = ae(ReR) = aeR = aR$. Since $aSR \subset JR \ll R$, $aSR \ll R$, so $aR \ll R$. Then by Lemma 3.1, $lr(a) = Ra$. It follows that $R\varphi(a) \subset lr(\varphi(a)) \subset lr(a) = Ra$. Then $\varphi(a) = e\varphi(a)$. Since $\varphi(a) = 1_R\varphi(a) \in R\varphi(a) \subset Ra$, $\varphi(a) \in Ra$ so $e\varphi(a) \in eRa$. Then $\varphi(a) = e\varphi(a) \in eRa = eRea = (eRe)a$, so $\varphi(a) = sa$ for some $s \in S$. Define $\hat{\varphi} : S \to S$ by $\hat{\varphi}(t) = st$ for every $t \in S$. Let $t_1, t_2 \in S$ such that $t_1 = t_2$. Then $st_1 = st_2$. Hence $\hat{\varphi}(t_1) = st_1 = st_2 = \hat{\varphi}(t_2)$. This shows that $\hat{\varphi}$ is well-defined. Let $t_1, t_2 \in S$ and $t \in S$. Then $\hat{\varphi}(t_1t + t_2) = s(t_1t + t_2) = st_1t + st_2 = \hat{\varphi}(t_1)t + \hat{\varphi}(t_2)$. This shows that $\hat{\varphi}$ is S-homomorphism. To show that $\varphi = \hat{\varphi}\, i$. Let $at \in aS$. Then $\varphi(at) = \varphi(a)t = \text{sat} = \hat{\varphi}(a)t = \hat{\varphi}(at) = \hat{\varphi}\, i(at)$. Hence eRe is right SP-injective.

□

3.2.4 Theorem. Let R be right *SP-injective,* $a \in R$ and $b \in J(R)$.

(1) If bR embeds in aR, then Rb is an image of Ra.

(2) If aR is an image of bR, then Ra embeds in Rb.

(3) If $bR \cong aR$, then $Ra \cong Rb$.

Proof. (1) Let $f : bR \to aR$ be an R-monomorphism. Since R is right SP-injective, there exists an R-homomorphism $\hat{f} : R \to R$ such that $\iota_2 f = \hat{f}\iota_1$ where $\iota_1 : bR \to R$ and $\iota_2 : aR \to R$ are the inclusion maps. Define $\sigma : Ra \to Rb$ by $\sigma(sa) = s\hat{f}(b)$ for every $s \in R$. If $sa = 0$, then $\sigma(sa) = s\hat{f}(b) = sf(b) \in s(aR) = (sa)R = 0$. To show that $Im(\hat{f}b) \subset Im(a)$. This shows that σ is well-defined. To show that σ is a left R-homomorphism. Let $s_1(a), s_2(a) \in Ra$ and $v \in R$. Then $\sigma(vs_1a + s_2a) = \sigma((vs_1 + s_2)a) = (vs_1+s_2)\hat{f}b = vs_1\hat{f}b + s_2\hat{f}b = v(s_1\hat{f}b) + s_2\hat{f}b = v\sigma(s_1a) + \sigma(s_2a)$. To show that σ is an R-epimorphism. Let $kb \in Rb$. To show that $r(\hat{f}(b)) \subset r(b)$. Let $x \in r(\hat{f}(b)$. Then $\hat{f}(b(x)) = 0$, so $f(b(x)) = \hat{f}(b(x)) = 0$. Since f is monic, $bx = 0$. Then $x \in r(b)$ and hence $lr(b) \subset lr(\hat{f}(b))$. Since $bR \ll R$ and $\hat{f} : R \to R$ is an R-homomorphism, $\hat{f}(b)R \ll R$ by Proposition 2.2.4. Since R is SP-injective, $Rb \subset R\hat{f}b$ by Lemma 3.2.1. Then $b = 1 \cdot b = s\hat{f}b$ for some $s \in R$. Hence there exists $ksa \in Ra$ such that $kb = \sigma(ksa)$.

(2) Let $f : bR \to aR$ be an R-epimorphism. Since R is SP-injective, there exists an R-homomorphism $\hat{f} : R \to R$ such that $\iota_2 f = \hat{f}\iota_1$ where $\iota_1 : bR \to R$ and $\iota_2 : aR \to R$ are the inclusion maps. Define $\sigma : Ra \to Rb$ by $\sigma(sa) = s\hat{f}(bx)$ for every $s \in R$. It is clear that σ is a left R-homomorphism. Let $sa \in Ker(\sigma)$. Then $0 = \sigma(sa) = s\hat{f}(bx) = sf(bx) = sa = 0$.

(3) Follows from (1) and (2) □

Following[1], a ring is R *semiprimitive* in case $J(R) = 0$.

3.2.5 Proposition. The following conditions are equivalent for a ring *R:*

(1) *R is semiprimitive*.

(2) Every right *R-module* is *SP-injective.*

(3) Every principal right ideal is *SP-injective.*

Proof. We only prove the right side, the left side is analogously. It is ovious that (1) $\Rightarrow$ (2) $\Rightarrow$ (3)**.** We show (3) $\Rightarrow$ (1). Suppose $J \neq 0$. Then there exists a nonzero element $a \in J(R)$.Then by assumption, the inclusion map from aR to R is split. Then aR is direct summand of R so $aR = 0$ which is contradiction. □

3.2.6 Theorem. The following conditions are equivalent for a ring *R:*

(1) Every small and principal right ideal of *R* is *projective.*

(2) Every factor module of an *SP-injective* module is *SP-injective.*

(3) Every factor module of an injective *R*-module is *SP-injective.*

Proof. (1) $\Rightarrow$ (2) Let M be an *SP*-injective module, X a submodule of M. To show that M/X is an *SP*-injective. Let $a \in J(R)$ and let $\varphi : aR \to M/X$ be an R-homomorphism. Since aR is projective, there exists an R-homomorphism $\alpha : aR \to M$ such that $\varphi = \eta\alpha$ where $\eta : M \to M/X$ is the natural R-epimorphism. Since M is *SP*-injective, there exists an R-homomorphism $\beta : R \to M$ such that $\alpha = \beta\iota$ where $\iota : aR \to R$ is the inclusion map. Then $\varphi = \eta\alpha = \eta\beta\iota$. Therefore $\eta\beta$ is an extension of φ . Thus M/X is an *SP*-injective.

(2) $\Rightarrow$ (3) Let M be an injective R-module and X be a submodule of M. It is clear that an injective R-module is an *SP*-injective module, so M is *SP*-injective. Then by (2), M/X is an *SP*-injective.

(3) $\Rightarrow$ (1) Let $aR \ll R$, $\gamma : A \to B$ be an R-epimorphism and let $\varphi : aR \to B$ be an R-homomorphism. Let E be an injective R-module and embed A in E by Proposition 2.5.4. Since γ is an R-epimorphism, by Proposition 2.4.4, there exists an R-isomorphism $\sigma : A/Ker(\gamma) \to B$ such that $\gamma = \sigma\eta_1$ where $\eta_1 : A \to A/Ker(\gamma)$ is the natural R-epimorphism. Then by Proposition 2.1.15, we have $\sigma^{-1} : B \to A/Ker(\gamma)$ is an R-isomorphism, so $B \cong A/Ker(\gamma)$ and $A/Ker(\gamma)$ is a submodule of $E/Ker(\gamma)$. By

assumption, there exists an R-homomorphism $\hat{\varphi}:M \rightarrow E/Ker(\gamma)$ such that $\iota_1\sigma^{-1}\varphi = \hat{\varphi}\iota_2$ where $\iota_1 : A/Ker(\gamma) \rightarrow E/Ker(\gamma)$ and $\iota_2 : aR \rightarrow R$ are the inclusion maps. Since R is projective, there exists an R-homomorphism $\beta : R \rightarrow E$ such that $\hat{\varphi} = \eta_2\beta$ where $\eta_2 : E \rightarrow E/Ker(\gamma)$ is the natural R-epimorphism. Then $\hat{\varphi}\iota_2 = \eta_2\beta\iota_2$. Hence $\iota_1\sigma^{-1}\varphi = \hat{\varphi}\iota_2 = \eta_2\beta\iota_2$. It follows that $\iota_1\sigma^{-1}\varphi = \eta_2\beta\iota_2$. To show that $\beta(aR) \subset A$. Let $a(r) \in a(R)$. Then $\iota_1\sigma^{-1}\varphi(a(r)) = \eta_2\beta\iota_2(a(r)) = \eta_2\beta(a(r)) = \eta_2(\beta(a(r))) = \beta(a(r)) + Ker(\gamma)$. Hence $\iota_1\sigma^{-1}\varphi(a(r)) = \sigma^{-1}\varphi(a(r)) = a + Ker(\gamma)$ for some $a \in A$, so $\beta(a(r)) + Ker(\gamma) = a + Ker(\gamma)$. Thus $\beta(a(r)) - a \in Ker(\gamma)$. It follows that $\beta(a(r)) = (\beta(a(r)) - a) + a \in Ker(\gamma) + A = A$. To show that $\varphi = \gamma\beta$. Let $a(r) \in a(R)$. Then $\iota_1\sigma^{-1}\varphi(a(r)) = \sigma^{-1}\varphi(a(r)) = \eta_2\beta\iota_2(a(r)) = \eta_2\beta(a(r))$. Hence $\iota_1\sigma^{-1}\varphi(a(r)) = \eta_2\beta(a(r)) = \beta(a(r)) + Ker(\gamma)$, so $\iota_1\sigma^{-1}\varphi(a(r)) = \beta(a(r)) + Ker(\gamma)$. Since γ is an R-epimorphism, $\varphi(a(r)) = \gamma(a)$ for some $a \in A$. Thus $\iota_1\sigma^{-1}\varphi(a(r)) = \iota_1\sigma^{-1}\gamma(a) = \sigma^{-1}\gamma(a) = \eta_1(a) = a + Ker(\gamma)$. It follows that $\beta(a(r)) + Ker(\gamma) = a + Ker(\gamma)$. Then $\beta(a(r)) - a \in Ker(\gamma)$. Hence $\gamma(\beta(a(r)) - a) = 0$, so $\gamma\beta(a(r)) = \gamma(a) = \varphi(a(r))$. Thus $\gamma\beta(a(r)) = \varphi(a(r))$. This shows that β lifts φ. □

3.2.7 Proposition. Let R be right *SP-injective* and $b_i \in J(R)$, $(1 \leq i \leq n)$.

(1) If $Rb_1 \oplus \ldots \oplus Rb_n$ is direct, then any R-homomorphism $\alpha : b_1R + \ldots + b_nR \rightarrow R$ can be extended to R.

(2) If $b_1R \oplus \ldots \oplus b_nR$ is direct, then $R(b_1 + \ldots + b_n) = Rb_1 + \ldots + Rb_n$.

Proof. (1) Let $Rb_1 \oplus \ldots \oplus Rb_n$ is direct and let $\alpha : b_1R + \ldots + b_nR \rightarrow R$ be an R-homomorphism. Since R is SP-injective, for each i, $1 \leq i \leq n$, there exists an R-homomorphism $\varphi_i : R \rightarrow R$ such that $\alpha(b_ir) = \varphi_i(b_ir)$ for every $r \in R$. Since $b_i(R) \ll R$ for each $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} b_i(R) \ll R$ by Proposition 2.2.3(2), and we have $(\sum_{i=1}^{n} b_i)(R) \subset \sum_{i=1}^{n} b_i(R)$ which implies $(\sum_{i=1}^{n} b_i)(R) \ll R$ by Proposition 2.2.3(1). Since R is SP-injective, there exists an R-homomorphism $\varphi : R \rightarrow R$ such that, for any $r \in R$, $\varphi(\sum_{i=1}^{n} b_i)(r) = \alpha(\sum_{i=1}^{n} b_i)(r)$. To show that $\sum_{i=1}^{n} \varphi(b_i) = \sum_{i=1}^{n} \varphi_i(b_i)$. Let $r \in R$.

Then $\sum_{i=1}^{n}\varphi_i b_i(r)$ = $\varphi_1 b_1(r) + \varphi_2 b_2(r) + \ldots + \varphi_n b_n(r) = \alpha b_1(r) + \alpha b_2(r) + \ldots + \alpha b_n(r)$ $= (\alpha b_1 + \alpha b_2 + \ldots + \alpha b_n)(r) = \alpha(b_1 + b_2 + \ldots + b_n)(r) = \alpha(\sum_{i=1}^{n} b_i)(R) = \varphi(\sum_{i=1}^{n} b_i)(R)$ $= \varphi(b_1 + b_2 + \ldots + b_n)(r) = (\varphi b_1 + \varphi b_2 + \ldots + \varphi b_n)(r) = \varphi b_1(r) + \varphi b_2(r) + \ldots + \varphi b_n(r) =$ $\sum_{i=1}^{n}\varphi b_i(r)$. This shows that $\sum_{i=1}^{n}\varphi(b_i) = \sum_{i=1}^{n}\varphi_i(b_i)$. Then $(\varphi_1 b_1 - \varphi b_1) + (\varphi_2 b_2 - \varphi b_2) + \ldots + (\varphi_n b_n - \varphi b_n) = 0$. Thus $(\varphi_1 - \varphi)b_1 + (\varphi_2 - \varphi)b_2 + \ldots + (\varphi_n - \varphi)b_n = 0$. Since $Rb_1 \oplus Rb_2 \oplus \ldots \oplus Rb_n$ is direct, $(\varphi_1 - \varphi) = (\varphi_2 - \varphi) = (\varphi_n - \varphi) = 0$. Then by Proposition 2.6.8, $(\varphi_1 - \varphi)b_1 = (\varphi_2 - \varphi)b_2 = \ldots = (\varphi_n - \varphi)b_n = 0$. Hence $(\varphi_i - \varphi)b_i = 0$, for all $1 \le i \le n$. Thus $\varphi_i(b_i) = \varphi(b_i)$, for all $1 \le i \le n$. To show that $\alpha = \varphi\iota$. Let $b_1(x_1) + b_2(x_2) + \ldots + b_n(x_n) \in b_1(R) + b_2(R) + \ldots + b_n(R)$. Then $\alpha(b_1(x_1) + b_2(x_2) + \ldots + b_n(x_n)) = \alpha b_1(x_1) + \alpha b_2(x_2) + \ldots + \alpha b_n(x_n) = \varphi_1 b_1(x_1) + \varphi_2 b_2(x_2) + \ldots + \varphi_n b_n(x_n) = \varphi b_1(x_1) + \varphi b_2(x_2) + \ldots + \varphi b_n(x_n) = \varphi(b_1(x_1) + b_2(x_2) + \ldots + b_n(x_n)) = \varphi\iota(b_1(x_1) + b_2(x_2) + \ldots + b_n(x_n))$. Hence $\alpha(b_1(x_1) + b_2(x_2) + \ldots + b_n(x_n)) = \varphi\iota(b_1(x_1) + b_2(x_2) + \ldots + b_n(x_n))$. This shows that φ is an extension of α.

(2) ($\supset$) Let $\alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_n b_n \in Rb_1 + Rb_2 + \ldots + Rb_n$. To show that $\alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_n b_n \in R(b_1 + b_2 + \ldots + b_n)$. For each i, define $\varphi_i : (b_1 + b_2 + \ldots + b_n)R \to R$ by $\varphi_i((b_1 + b_2 + \ldots + b_n)r) = b_i r$ for every $r \in R$. Let $0 = (b_1 + b_2 + \ldots + b_n)(r) \in (b_1 + b_2 + \ldots + b_n)R$. Then $b_1(r) + b_2(r) + \ldots + b_n(r) = (b_1 + b_2 + \ldots + b_n)R = 0$. Since $b_1R \oplus b_2R \oplus \ldots \oplus b_nR$ is direct, $b_1r = b_2r = \ldots = b_nr = 0$ so $b_ir = 0$. This shows that φ_i is well-defined. Let $(b_1 + b_2 + \ldots + b_n)r_1, (b_1 + b_2 + \ldots + b_n)r_2 \in (b_1 + b_2 + \ldots + b_n)R$. Then $\varphi_i((b_1 + b_2 + \ldots + b_n)(r_1)r + (b_1 + b_2 + \ldots + b_n)(r_2)) = \varphi_i((b_1 + b_2 + \ldots + b_n)(r_1r + r_2)) = b_i(r_1r + r_2) = b_i(r_1r) + b_i(r_2) = b_i(r_1)r + b_i(r_2) = \varphi_i((b_1 + b_2 + \ldots + b_n)(r_1))r + \varphi_i((b_1 + b_2 + \ldots + b_n)(r_2))$. This shows that φ_i is an R-homomorphism. By the similar proof of (1) we have $(b_1 + b_2 + \ldots + b_n)R \ll R$. Since R is SP-injective, there exists an R-homomorphism $\hat{\varphi}_i : R \to R$ such that $\varphi_i = \hat{\varphi}_i\,\iota$ where $\iota : (b_1 + b_2 + \ldots + b_n)R \to R$ is the inclusion map. Then $b_i = \varphi_i(b_1 + b_2 + \ldots + b_n) = \hat{\varphi}_i(b_1 + b_2 + \ldots + b_n) \in R(b_1 + b_2 + \ldots + b_n)$. Hence $\alpha_i b_i = \alpha_i\hat{\varphi}_i(b_1 + b_2 + \ldots + b_n) \in R(b_1 + b_2 + \ldots + b_n)$ so $\alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_n b_n = \alpha_1\hat{\varphi}_1(b_1 + b_2 + \ldots + b_n) + \alpha_2\hat{\varphi}_2(b_1 + b_2 + \ldots + b_n) + \ldots + \alpha_n\hat{\varphi}_n(b_1 + b_2 + \ldots + b_n) = (\alpha_1\hat{\varphi}_1 + \alpha_2\hat{\varphi}_2 + \ldots + \alpha_n\hat{\varphi}_n)(b_1 + b_2 + \ldots + b_n) \in R(b_1 + b_2 +$

$\ldots + b_n$). ($\subset$) Let $\alpha(b_1 + b_2 + \ldots + b_n) \in R(b_1 + b_2 + \ldots + b_n)$. Then $\alpha(b_1 + b_2 + \ldots + b_n) = \alpha b_1 + \alpha b_2 + \ldots + \alpha b_n \in Rb_1 +\ldots+ Rb_n$. □

3.2.8 Proposition. Let R be right *SP-injective* and $B_1 \oplus \ldots \oplus B_n$ a direct sum of small(two – side) ideals of R. Then for any fully invariant ideal A of R, we have

$$A \cap (B_1 \oplus \ldots \oplus B_n) = (A \cap B_1) \oplus \ldots \oplus (A \cap B_n).$$

Proof. ($\supset$) Since $A \cap B_i \subset A \cap (B_1 \oplus \ldots \oplus B_n)$ for each $i = 1, 2, \ldots, n$, we have $(A \cap B_1) \oplus \ldots \oplus (A \cap B_n) \subset A \cap (B_1 \oplus \ldots \oplus B_n)$. ($\subset$) Let $a = \sum_{i=1}^{n} b_i \in A \cap (B_1 \oplus \ldots \oplus B_n)$. To show that $\sum_{i=1}^{n} b_i \in (A \cap B_1) \oplus \ldots \oplus (A \cap B_n)$. Let $\pi_k : \oplus_{i=1}^{n} b_i R \to b_k R$ be the projection map. Since for each i, $(1 \le i \le n)$, $Rb_i \subset B_i$. Thus $\oplus_{i=1}^{n} Rb_i$ is direct. By Proposition 3.2.7, π_k has an extension $\hat{\pi}_k : R \to b_k R$ such that $\pi_k = \hat{\pi}_k \iota$ where $\iota : \oplus_{i=1}^{n} b_i R \to R$ is the inclusion map. Let $r_i \in R$. Then $b_i = \pi_i \sum_{i=1}^{n} b_i = \hat{\pi}_i \iota \sum_{i=1}^{n} b_i = \hat{\pi}_i (\sum_{i=1}^{n} b_i) = \hat{\pi}_i (a) \in A \cap B_i$. Hence $\sum_{i=1}^{n} b_i = b_1 + b_2 + \ldots + b_n \in A \cap B_1 \oplus A \cap B_2 \oplus \ldots \oplus A \cap B_n$. □

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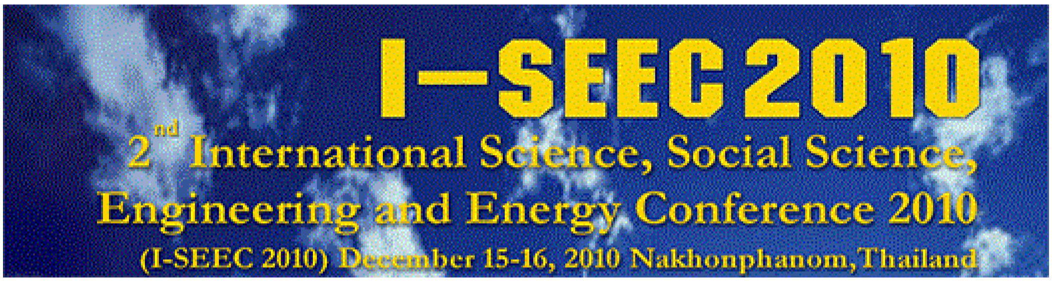
APPENDIX



APPENDIX A

Conference Proceeding

Paper Title "On Small Principally injective Rings"

The 2nd International Science, Social Science, Engineering and Energy

Conference 2010

At   Nakhonphanom River View Hotel

December 15-16, 2010

Schedule of I-SEEC 2010

Date/Time	International Science, Social-Science, Engineering and Energy Conference (I-SEEC 2010) 15-16 December, 2010 Nakhonphanom,Thailand. Rajamangala University of Technology Isan Sakon Nakhon Campus, Sakon Nakhon, Thailand. December 15, 2010			
8.00-12.00	**Registration** (Important notice: Please bring your registration receipt to the registration desk to verify your registration status. After doing so, you may submit your full paper: 3 hard copies + a CD containing electronic file of you manuscript.) We encourage EVERYONE to registration or verify your registration TODAY in order to avoid a long-waiting line due to a large number of conference participants. * Manuscript must be handed in before noon,			
9.30-10.00	**Opening Ceremony** *MC: Dr.Ada Raimaturapong*			
	ROOM1	ROOM2	ROOM3	ROOM4
10.00-10.45	**Plenary Lecture I :** 'Nutrition in health and disease in dogs' *Prof. Dr. Anton C.Beynen, Utrecht University, Netherlands.*			
10.45-11.30	**Plenary Lecture II :** 'A PANDA Ring Resonator Design and Applications' *Prof. Dr. Preecha Yupapin, Head of ARCP, KMITL, Thailand.*			
11.30-12.15	**Plenary Lecture III:** 'Precision force measurement using the Levitation Mass Method (LMM)' *Prof. Dr. Yusaku Fujii, Gunma University, Japan.*			

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12.15-13.00	Lunch			
Time/Room	ROOM1	ROOM2	ROOM3	ROOM4
13.00-13.30	**Invited Speaker:** Functional Hybrid Optical Nanocomposites Based on Metal OxideOrganic-Nanostructure Materials **Session Chair:** ***Assoc.Prof.Dr.Wisanu Pecharapa***	**Invited Speaker:** Research on Heat Transfer Enhancement in Refrigeration and Air Conditioning **Session Chair:** *Prof.Dr.Somchai Wongwises*	**Invited Speaker:** **AS0014:** Effects of Pen Types and Sizes on Production of DYL Cross-Bred Pigs **Session Chair:** *Assist.Prof.Dr.Jakrit Yaeram*	**Invited Speaker:** **SS0010:** Rhythms of Village Life in Psychology of Rice Worship Ceremony **Session Chair:** *Dr.Kanopporn Wonggarasin*
13.30-13.45	**OS0014:** Ozone-Induced Optical Density Change of NiO Thin Films and Their Applicability as Neutral Optical Density Filter *R. Noonuruk* *N. Wongpisutpaisan* *P. Mukdacharoenchai* *W. Techitdheera* *W. Pecharapa*	**EE0012:** A Biomarkers Study Trace Metal Elements in Paphia Undulate Shell for Assessing Pollution of Coastal Area *N. Juncharoenwongsa* *W. Siriprom* *J. Kaewkhao* *A. Choeysuppaket* *P. Limsuwan* *K. Phachana*	**AS0002:** Portable Electronic Nose and its Applications in Vineyard *K. Teerakiat* *L. Panida*	**SS0001:** Organizational Learning Sustaining the Competitive *P.Chanthima* *P.Parnisara* *P.Darika*
13.45-14.00	**OS0015:** All Optical Half AdderSubtractor using Dark-bright Soliton Conversion Control Conversion Control *T.Sappasit* *P.Yupapin*	**EE0013:** A Numerical Computation of Water Quality Measurement in a Uniform Channel Using a Finite Difference Method *P.Nopparat* *D.Rujira*	**AS0003:** Effect assessment of carcasses washing on the prevalence of Salmonella contamination at different stages of poultry slaughter *Sana,M.j.*	**SS0003:** The Influence of Organizational Support, Market Turbulence and Business Strategy on Market Orientation and New Product Activity *P.Parnisara* *P.Chanthima* *P.Darika*

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Time/Room	ROOM1	ROOM2	ROOM3	ROOM4
14.00-14.15	**OS0018:** Determination of the thickness and optical constants of ZrO2 by spectroscopic ellipsometry and spectrophotometric method *R. Yusoh* *M. Horprathum* *P. Eiamchai* *S. Chanyawadee* *K. Aiempanakit*	**EE0014:** A Non-dimensional Form of Hydrodynamic Model with Variable coefficients in a Uniform Reservoir Using *P.Nopparat* *S.Chunya*	**AS0004:** Karyological and Randomly Amplified Polymorphic DNA-Polymerase Chain Reaction Studies in Barbodes spp. in Northeastern Thailand *P.Keeravit* *T.Bungorn*	**SS0005:** Antecedents and Consequence of Organizational Commitment *P.Surasit* *Y.Nikorn* *P.Ampasri*
14.15-14.30	**OS0046:** Design and Preparation of Synthetic Hydrogels Via Photopolymerisation for Biomedical Use as Wound Dressings *W.Chinanat*	**EE0016:** Characterization of PVA-Chitosan Nanofibers Prepared by Electrospinning *K. Paipitak* *T. Pornpra* *P. Mongkontalang* *W. Techitdheera* *W. Pecharapa*	**AS0005:** Influence of Graded Level Protein of Cassava Root Meal Based Diets on Performance of Growing Pigs V.Ratchaneevan *B.Wandee* *K.Nukon* *M.Chartchai* *B.Suttipong*	**SS0008:** Guideline on Community Radio Management for Social Civil Uttaradit Community Strength *T.Radee* *C.Pannita* *Y.Sirikran*
14.30-14.45	**OS0006:** A Novel Dynamic Optical Tweezers Array Generation Using Dark Soliton Control Within double AddDrop Multiplayer *N. Pornsuwancharoen* *V. Thanyaphirak* *U. Punmeekeaw* *P.P. Yupapin*	**EE0018:** Partial Discharge Monitoring System for Power Distribution Transformers as a Basis Risk Assessment Insulation Evaluation *A.Promsak* *P.Winai* *P.Boonyang*	**AS0006:** Effects of a hCG Administration on d 5 after a Timed Artificial Insemination on the Conception Rate of Postpartum Beef Cows in Small Holder Farmers *S. Guntaprom* *J. Yaeram* *J. Jugsumrit* *W. Pratoompol* *C. Ampom*	**SS0012:** Perceptions of Professional Ethics in Thailand Conceptual Paper *K.Thanit* *B.Sumintorn*
14.45-15.00	**Coffee / Tea Break**			

Time/Room	ROOM1	ROOM2	ROOM3	ROOM4
15.00-15.30	**Invited Speaker:** SAFETY INSPECTION STRATEGY FOR EARTH MBANKMENT DAMS USING FULLY DISTRIBUTED SENSING by Prof. Dr. P.Y. Zhu **Session Chair:** *Assoc.Prof.Dr.Somsak Mitatha*	**Invited Speaker:** Future Global Visions of Engineering Education Invited Speaker by Assoc.Dr.Wisuit Sunthonkanokpong **Session Chair:** *Assoc.Dr.Wisuit Sunthonkanokpong*	**Invited Speaker:** **AS0024:** Effect of fat type on feed intake rumen fermentation and nutrient digestibility in beef cattle **Session Chair:** Assist. Prof. *Assist.Prof. Dr. Chalermpon Yuangklang*	**Invited Speaker:** **BU0008:** Perceived Organizational Support on organizational commitment of Thai employees in rajabhat universities in the northern group **Session Chair:** *Dr.Sumintorn Baotham*
15.30-15.45	**Invited Speaker:** Development of lead free radiation shielding glass experimental and theoretical approach by Dr. Jakrapong Kaewkhao	**EE0023:** Development of Electrical Transient Modeling and Simulation for Electric Distribution Systems *B.Banyat* *S.Chitchai*	**AS0009:** Biocontrol of Seed-Borne Pathogenic Fungi of Cabbage Seedling by Curcuma longa Extract *C.Piyanan*	**BU0001:** Audit Independence in Appearance in Thailand Conceptual Paper *K.Payom* *B.Sumintorn*
15.45-16.00	**EC0004:** Theoretical calculation of optical absorption spectrum for Armchair graphene nanoribbon by first principle calculation *E. Ahmadi* *A. Asgari*	**EE0029:** Synthesis on the Nanoparticle of LaCoO3 Thermoelectric Material *K.PASARA* *T.CHANCHANA* *S.TOSAWAT*	**AS0012:** Frog Culture Development in Accordance with Sufficiency Economy Philosophy *T.Thongyoon* *Y.Jaknit* *P.Wasan* *R.Wutti* *P.chaisongkram* *Poogingngern*	**BU0002:** Antecedent and Consequence of Job Satisfaction and Organizational Commitment of Thai Employees in RMUTT Conceptual Paper *B.Sumintorn*

Time/Room	ROOM1	ROOM2	ROOM3	ROOM4
16.00.16.15	**EC0007:** *The Wireless Electrical Load Automation Control System for Electrical Energy Saving* *N.Sarawut*	**NC0008:** Image Recorder Server with IP Camera and Pocket PC *N. Boonma* *A. Sangthong* *S. Mitatha and C. Vongchumyen*	**AS0013:** Effect of Biologically Fermented Herbs Juices and Probiotics Supplementation in Organic Dairy Cows *S.Nunthiya* *K.Jumrian* *P.Chaiya* *N.Nuttawut* *T.Nimlamai*	**BU0003:** Effect of SCF from Operating Activities Format on Lenders' Decision Conceptual Paper *Y.Arpa* *B.Sumintorn*
16.15-16.30	**EC0011:** Small – Signal Model of Series – Parallel Resonant *K.Chengchan*	**NC0009:** SMS Information Display Board *A. Tanadumrongpattana* *A. Suethakorn* *S. Mitatha and* *C. Vongchumyen*	**AS0014:** Effects of Pen Types and Sizes on Production of DYL Cross-Bred Pigs *Y.Jakrit* *P.Wasun* *S.Jirasak* *D.Sarawut*	**BU0005:** Innovation Capability, Market Orientation Constructs and Business Performance in Thailand Conceptual Paper *W.Duangrudee* *B.Sumintorn* *B.Savitee*
16.30-16.45	**MT0002:** On Small Principally Injective Rings *K. Amnuaykarn* *S. Wongwai*	**NC0010:** Wireless Traffic Light Controller *K. Thatsanavipas* *N. Ponganunchoke* *S. Mitatha and C. Vongchumyen*	**AS0015:** Karyological and RAPD-PCR Studies in *Barbodes spp* *P.Keeravit* *T.Bungorn*	**BU0006:** National Culture on Accounting Values Gray s Constructs Conceptual Paper Phase 1 *T.Wimolyai* *B.Sumintorn* *K.Lertluk*
16.30-17.00	**Poster Board Part I**			
17.00-18.00	**Presentation Poster**			
18.30-22.00	**Welcome Party. (swimming pool-side, beer garden, ground floor, khong river)**			

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December 16, 2010				
Time/Room	**ROOM1**	**ROOM2**	**ROOM3**	**ROOM4**
9.00-10.00	Poster Sesion Part2 / Free Time Presentation Poster	**Plenary Lecture IV:** Prof. Dr. Koichi KAKU, AFFFRC, Japan. 'Global warming and climate change of Asian Countries including Japanese domestic GHG emission from the standpoint of Clean Developing Mechanism (CDM) for Green House Gas (GHG) reduction in the Field of Agriculture'	**Plenary Lecture V:** Dr. Sudarath Sakhunkhu, RMUTI, Thailand. 'Yield Trial of 6 Varieties of Azukibean in Sakhon-Nakhon'	
10.00-10.30	Poster Sesion Part2 / Free Time Presentation Poster	**Invited Speaker: EC0008:** Harmonic Suppression Improvement of Microstrip Open Loop Ring Resonator Bandpass Filter **Session Chair:** *Dr.Ravee Phromloungsri*	**Invited Speaker:AS0018:** Effect of Location for raised on Growth Performance and Carcass Quality of Kadon Pig **Session Chair:** *Assits.Prof.Dr.Kraisit Vasupen*	**Invited Speaker:BU0004:** Effects of Organizational Alliance Success on Performance of Hotel Industry in Thailand **Session Chair:** *Dr.Chanthima Phromket*
10.30-10.45	**Coffee / Tea Break**			
10.45-11.00	Poster Sesion Part3 / Free Time Presentation Poster	**EE0031:** The Effect of Jumbo Compact Fluorescent Lamp Energy Saving Lamp on the Electrical Energy Saving and Harmonics Noise *U.Chutipon*	**AS0019:** Interactive effects of the feeding of cassava leaves and calcium level on feed intake and macronutrient digestion in beef cattle *J. Khotsakdee* *N. Opatawong* *K. Vasupen* *S. Bureenok* *S. Wongsuthavas* *P. Panyakaew* *C. Yuangklang*	**BU0007:** National Culture on Accounting Values Gray s Conservatism and Secrecy Hypothesis Conceptual Paper Phase 2 *H.Wilaiporn* *B.Sumintorn* *K.Nuchnapar*

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Time/Room	ROOM1	ROOM2	ROOM3	ROOM4
11.00-11.15		EE0032:Side Effect of T5 Ballast on the Conducted Electromagnetic Interference or EMI and Harmonics Noise U.Chutipon	AS0021:In vitro gas production measurements to evaluate different fat sources based on urea-treated rice straw as main roughage source J. Khotsakdee P. Hunghuan K. Vasupen S. Bureenok S. Wongsuthavas P. Panyakaew C. Yuangklang	BU0009:Professional_Ethic_on_Auditor_and_Public_Perception_in_Thailand_(1) K.Thanit B.Sumintorn
11.15-11.30		EE0036:The Effect of Power Supply of LED Lamp K.Patiphan K.Chengchan U.Chutipon	AS0024:Effect of fat type on feed intake rumen fermentation and nutrient digestibility in beef cattle K. Kongweha K. Vasupen P. Paengkoum S. Bureenok S. Wongsuthavas C. Yuangklang	
11.30-11.45			AS0010:Growth rate, heat tolerance, dressing percentage and rib eye area of Brahman and Angus crossbred steers under conventional fattening fed rice straw based diet in Thailand C. Promkot P. Pornanake	
11.30-11.45		-		
12.00-13.00	Lunch			

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Contents	Page.

Energy and Environment

Electronic and Communication

Network Technologies and Computation Intelligence

Optical Science and Technology

Network Technologies and Computation Intelligence

Social Science

Business

Others

On Small Principally Injective Rings

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Abstract

Let R be a ring. A right R- module M is called *small principally injective* (briefly, $SP-$*injective*) if, every R-homomorphism from a small and principal right ideal aR to M can be extended to an R-homomorphism from R to M [11]. A ring R is called right $SP-$injective if, R_R is SP-injective. In this paper, we give some characterizations and properties of small principally injective modules and small principally injective rings.

Keywords: Professional Ethics.

1. Introduction

Throughout this paper, R will be an associative ring with identity and all modules are unitary right R-modules. For right R-modules M and N, $\mathrm{Hom}_R(M,N)$ denotes the set of all R-homomorphisms from M to N and $S = End_R(M)$ denotes the endomorphism ring of M. By notations, $N \subseteq^{\oplus} M$, $N \subseteq^{e} M$, and $N \ll M$ we mean that N is a direct summand, an essential submodule and a superfluous submodule of M, respectively.We denote the socle and the Jacobson radical of M by $Soc(M)$ and $J(M)$, respectively.

Let R be a ring. A right R-module M is called *principally injective* (or P-*injective*), if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$, where l and r are the left and right annihilators, respectively. This notion was introduced by Camillo [2] for commutative rings. In [7], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. They also continued to study rings with some other kind of injectivity, namely, mininjective rings [8]. A ring R is called *right mininjective* if every isomorphism between simple right ideals is given by left multiplication by an element of R. Equivalently, if kR is simple, $k \in R$, $lr(k) = Rk$. In [11], L.V. Thuyet, and T.C. Quynh, introduced a small principally module. A right R- module M is called *small principally injective (or $SP-$injective)* if, every R-homomorphism from a small and principal right ideal aR to M can be extended to an R-homomorphism from R to M. In this paper we also consider small principally injective modules and rings.

Following [1], a submodule K of a right R-module M is *superfluous* (or *small*) in M, abbreviated $K \ll M$, in case for every submodule L of M, $K + L = M$ implies $L = M$. It is clear that $aR \ll R$ if and only if $a \in J(R)$.

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2. SP-injective Modules

Definition 2.1. [11] Let R be a ring. A right R- module M is called *small principally injective* (briefly, *$SP-$injective*) if, every R-homomorphism from a small and principal right ideal aR to M can be extended to an R-homomorphism from R to M.

Lemma 2.2. *Let M be a right R-module. Then M is SP-injective if and only if each $a \in J(R)$, $l_M r_R(a) = Ma$.*

Proof. Clearly, $Ma \subset l_M r_R(a)$. Let $x \in l_M r_R(a)$. Define $\varphi : aR \to xR$ by $\varphi(ar) = xr$, for every $r \in R$. Since $r_R(a) \subset r_R(x)$, φ is well-defined so it is clear that φ is an $R-$homomorphism. Since M is SP-injective, there exists an $R-$homomorphism $\hat{\varphi} : R \to M$ such that $\hat{\varphi}\iota_2 = \iota_1\varphi$, where $\iota_1 : xR \to M$ and $\iota_2 : aR \to R$ are the inclusion maps. Then $x = \varphi(a) = \hat{\varphi}(1)a \in Ma$.

Conversely, let $a \in J(R)$, and let $\varphi : aR \to M$ be an $R-$homomorphism. Then $\varphi(a) \in l_M r_R(a)$, so by assumption, we have $\varphi(a) = xa$ for some $x \in M$. Define $\hat{\varphi} : R \to M$ by $\hat{\varphi}(r) = xr$ every $r \in R$. It is clear that $\hat{\varphi}$ is an R-homomorphism and is an extension of φ. □

Example 2.3. *Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field, and $M_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then M is $SP-$injective.*

Proof. It is clear that only $A = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is the nonzero small principal right ideal of R. Let $0 \neq a \in A$. Then $r_R(a) = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ so $l_M r_R(a) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. It is obvious that $Ma = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Then by Lemma 2.2, M is $SP-$injective. □

Proposition 2.4. *Let $\{M_i : i \in I\}$ be a family of right R-modules. Then the direct product $\prod_{i \in I} M_i$ is SP-injective if and only if each M_i is SP-injective.*

Proof. Let π_i and φ_i, for each $i \in I$, be the ith projection map and the ith injection map, respectively. We now let $i \in I$, $a \in J(R)$, and let $\varphi : aR \to M_i$ be an $R-$homomorphism. Then by assumption, there exists an $R-$homomorphism $\hat{\varphi} : R \to M_i$ such that $\hat{\varphi}\iota = \varphi_i\varphi$ where $\iota : aR \to R$ is the inclusion map. Thus $\varphi = \pi_i\hat{\varphi}\iota$. Conversely, let $a \in J(R)$ and $\varphi : aR \to \prod_{i \in I} M_i$ be an $R-$homomorphism. Then for each $i \in I$, there exists an $R-$homomorphism $\alpha_i : R \to M_i$ such that $\alpha_i\iota = \pi_i\varphi$ where $\iota : aR \to R$ is the inclusion map. Hence we obtain (product) $\hat{\varphi} : R \to \prod_{i \in I} M_i$ with $\pi_i\hat{\varphi} = \alpha_i$ and $\pi_i\hat{\varphi}\iota = \alpha_i\iota$ which implies $\hat{\varphi}\iota = \varphi$. □

Lemma 2.5. *Let M_i $(1 \leq i \leq n)$ be SP-injective modules. Then $\oplus_{i=1}^{n} M_i$ is SP-injective.*

Proof. It is enough to prove the result for $n = 2$. Let $a \in J(R)$ and $\varphi : aR \to M_1 \bigoplus M_2$ be an R-homomorphism. Since M_1 and M_2 are SP-injective, there exists an R-homomorphisms $\varphi_1 : R \to M_1$ and $\varphi_2 : R \to M_2$ such that $\varphi_1\iota = \pi_1\varphi$ and $\varphi_2\iota = \pi_2\varphi$ where π_1 and π_2 are the projection maps from $N_1 \bigoplus N_2$ to N_1 and N_2, respectively, and $\iota : aR \to R$ is the inclusion map. Set $\hat{\varphi} = \iota_1\varphi_1 + \iota_2\varphi_2 : R \to M_1 \bigoplus M_2$. Thus it is clear that $\hat{\varphi}$ extends φ. □

Lemma 2.6. *Any direct summand of an SP-injective module is again SP-injective.*

Proof. By definition. □

3. SP-injective Rings

If R_R is an SP-injective module, then we call R is a *right SP-injective ring*. In this section, we give some properties and characterizations of SP-injective rings.

The following lemma follows from Lemma 2.2.

Lemma 3.1. *Let R be a ring. Then R is right SP-injective if and only if each $a \in J(R)$, $lr(a) = Ra$.*

Theorem 3.2. *Let R be a right SP-injective ring. Then*

(1) *R is right mininjective.*

(2) *$lr(Ra) = Ra$, for any $a \in J(R)$.*

(3) *If $aR \oplus bR$ and $Ra \oplus Rb$ are both direct, $a, b \in J(R)$, then $l(a) + l(b) = R$.*

Proof. (1) Since every simple right ideal of R is either nilpotent or a direct summand of R [4, (10.22) Brauer's Lemma], each right $SP-$injective ring is right mininjective ring.

(2) Let $x \in lr(Ra)$. Define $\varphi : aR \to xR$ by $\varphi(ar) = xr$ for every $r \in R$. Since $ar = 0$ implies $xr = 0$, φ is well-defined. It is clear that φ is an $R-$homomorphism. Since R is right SP-injective, there exists an extension $\hat{\varphi} : R \to R$ of φ. Hence $x = \varphi(a) = \hat{\varphi}(1)a \in Ra$. This shows that $lr(Ra) \subset Ra$. The inclusion $Ra \subset lr(Ra)$ is always holds.

(3) Define $\varphi : (a+b)R \to R$ by $\varphi(a+b)r = br$ for every $r \in R$. If $(a+b)r = 0$, then $ar = br \in aR \cap bR = 0$ so $br = 0$. This shows that φ is well-defined. It is clear that φ is an $R-$homomorphism. Then there exists an extension $\hat{\varphi} : R \to R$ of φ. Hence $\hat{\varphi}(1)(a+b)$ $= \varphi(a+b) = b$ so $\hat{\varphi}(1)a = (1-\hat{\varphi}(1))b \in Ra \cap Rb = 0$. Then $\hat{\varphi}(1) \in l(a)$ and $(1-\hat{\varphi}(1)) \in l(b)$. Hence $1 = \hat{\varphi}(1) + (1-\hat{\varphi}(1)) \in l(a) + l(b)$. It follows that $l(a) + l(b) = R$. □

Proposition 3.3. *If R is right SP-injective, so is eRe for all $e^2 = e \in R$ satisfying $ReR = R$.*

Proof. Write $S = eRe$ and let $\varphi : aS \to S$ be an $S-$homomorphism, where $a \in J(S)$. Then $r(a) \subset r(\varphi(a))$ so $lr(\varphi(a)) \subset lr(a)$. Since $aSR = aeReR = aeR = aR$ and $aSR \ll R$, $aR \ll R$. Then by Lemma 3.1, $lr(a) = Ra$. It follows that $R\varphi(a) \subset lr(\varphi(a)) \subset lr(a) = Ra$. Then $\varphi(a) = e\varphi(a) \in eRa = (eRe)a = Sa$ so $\varphi(a) = sa$ where $s \in S$. Define $\hat{\varphi} : S \to S$ by $\hat{\varphi}(t) = st$, for every $t \in S$. It is clear that $\hat{\varphi}$ is an $S-$homomorphism. Then for each $at \in aS$, $\varphi(at) = sat = \hat{\varphi}(at)$. Hence eRe is right SP-injective. □

Theorem 3.4. *Let R be right SP-injective, $a \in R$ and $b \in J(R)$.*

(1) *If bR embeds in aR, then Rb is an image of Ra.*

(2) *If aR is an image of bR, then Ra embeds in Rb.*

(3) *If $bR \simeq aR$, then $Ra \simeq Rb$.*

Proof. (1) Let $f : bR \to aR$ be an $R-$monomorphism. Let $\iota_1 : bR \to R$ and $\iota_2 : aR \to R$ be the inclusion maps. Since R is right SP-injective, there exists an $R-$homomorphism $\widehat{f} : R \to R$ such that $\iota_2 f = \widehat{f}\iota_1$. Let $\sigma : Ra \to Rb$ defined by $\sigma(sa) = s\widehat{f}(b)$ for every $s \in R$. If $sa = 0$, then $\sigma(sa) = s\widehat{f}(b) = sf(b) \in s(aR) = (sa)R = 0$. This shows that σ is well-defined. It is clear that σ is an $R-$homomorphism. Note that $\widehat{f}(b)R \ll R$ by [**1**, Lemma 5.18]. For any $s \in R$, $\widehat{f}(b)s = 0$ implies $f(bs) = 0$ so $bs = 0$ because f is monic. Consequently, $r(\widehat{f}(b)) \subset r(b)$ and hence $lr((b) \subset lr(\widehat{f}(b))$. Then by Lemma 3.1, $Rb \subset R\widehat{f}(b)$. Thus $b \in R\widehat{f}(b)$ and so $b = s\widehat{f}(b) = \sigma(sa)$.

(2) By the same notations as in (1), let $f : bR \to aR$ be an $R-$epimorphism. Since R be right SP-injective, f can be extended to $\widehat{f} : R \to R$ such that $\iota_2 f = \widehat{f}\iota_1$. Write $a = f(bx) = \widehat{f}(bx), x \in R$. Define $\sigma : Ra \to Rb$ by $\sigma(sa) = s\widehat{f}(bx)$ for every $s \in R$. It is clear that σ is an R-homomorphism. If $sa \in Ker(\sigma)$, then $0 = \sigma(sa) = s\widehat{f}(bx) = sf(bx) = sa$. Hence σ is an $R-$monomorphism.

(3) Follows from (1) and (2).

Following [1], a ring is R *semiprimitive* in case $J(R) = 0$.

Proposition 3.5. *The following conditions are equivalent for a ring R:*

(1) *R is semiprimitive.*

(2) *Every right $R-$module is $SP-$injective.*

(3) *Every principal right ideal is $SP-$injective.*

Proof. (1)⇒(2)⇒(3) is clear.

(3)⇒(1) Suppose $J \neq 0$. Then there exists a nonzero element $a \in J(R)$. Then by assumption, the inclusion map from aR to R is split. Then aR is a direct summand of R so $aR = 0$ which is a contradiction. □

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