Free Vibration of Beams via Finite Difference Scheme

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Abstract

This paper aims to present the finite difference scheme tackling with the wave equation. Based on finite difference scheme, the dynamic equilibrium shapes and natural frequency of three types of beams (e.g. simple, fixed-fixed and cantilever beams) have been evaluated. For dynamic equilibrium, the explicit finite difference equations have been derived, the consistency of the scheme has been proved and the stability of the scheme has been determined. The natural frequencies are demonstrated by finding the lowest eigenvalue of each type of the beam. From the results, it is found that the accuracy of the results depend on the number of discretization for both time and space. However, the discretization of time and space must obey the stability condition which is $a = \frac{EIk^2}{mh^4} < \frac{1}{4}$.

Keywords : Finite difference, beam, eigenvalue, natural frequency

Introduction

In the real world problems, there are many problems that take the effect from vibration which comes from nature and human being. The well-known vibration from nature is the earthquake. This is the natural disaster which killed many people in the past. The proliferation of earthquake in many part of the world especially in Thailand is the signal to focus on the study of vibration of the structures for preventing the harm from the earthquake. Fig. 1a and 1b show the damage of structures due to earthquake.

1 a) 1 b)

Fig. 1 The damage of structures due to earthquake; a) 1952 Santa Babara earthquake; b) 1933 Long beach California earthquake.

There are also many types of vibration due to human being such as from explosion, running of cars or machines ect. Then the vibration of the structures should be investigated to find the behavior of vibration of each type of structural members. In this paper, the free vibration of three major types of beams (simple

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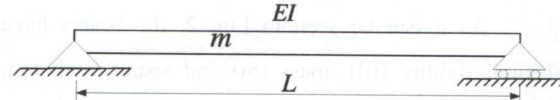
beam, fixed-fixed beam and cantilever beam) will be considered. This paper purposes to i) find the numerical scheme for studying the free vibration of three major types of beams (e.g. simple beam, fixed-fixed beam and cantilever beam) and ii)find the natural or fundamental frequency of these beams.

As aforementioned objectives, then the finite difference scheme for the problem is established. The results will be compared with exact solutions. Finally the physical behaviors of vibration of these beams will be investigated.

Statement of the problem and notations

This section the meaning of each type of beams and notations are considered. In Fig. 2, three types of beams are shown. Fig. 2a) is the simple beam; one end of beam is hinged and another end is placed on the roller support. The hinged end constrains the translation in both x and y directions. The roller support constrains only in y direction. However at the both ends, the beam can rotate freely about its axis. These properties of support ends are utilized as boundary conditions of the problems. For the simple beam, at hinged end there is no deflection and bending moment about this end is zero. Fig. 2b) is fixed-fixed beam; both ends are fixed then there is no translation and rotation about its axis. The boundary conditions of this type of beam are the deflections and rotations at its ends are zero. Fig. 2c) is the cantilever beam; one is fixed again translation and rotation another end is free end. Then the boundary conditions for this type of the beam are no deflection and slope at fixed end and there are no bending moment and shear force at free end. Table 1 shows the summary of boundary conditions for each type of the beams.



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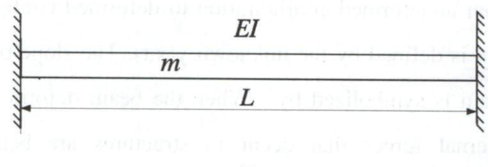

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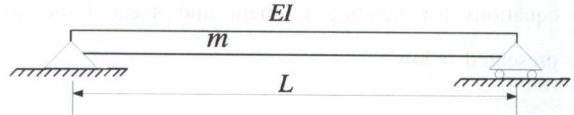

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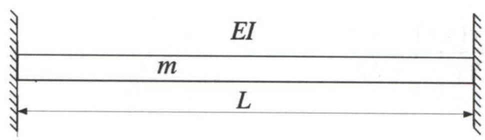

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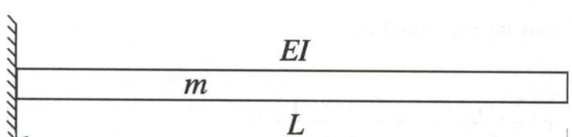

2c)

Fig. 2 Three types of beams; a) simple beam b) fixed-fixed beam c) cantilever beam

Table 1. Boundary conditions of the beams

Types of Beams	At $x=0$				At $x=L$			
	$y(0,t)$	$q(0,t)$	M	V	$y(0,t)$	$q(0,t)$	M	V
simple	0	น 0	0	น 0	0	น 0	0	น 0
fixed- fixed	0	0	น 0	น 0	0	0	น 0	น 0
cantilever	0	0	น 0	น 0	น 0	น 0	0	0

As it can be seen in Fig. 2, the beams have flexural rigidity (EI), mass (m) and span length (L). The parameters (EI, m and L) are defined as constants. The beams deform in y direction and then the measure from undeformed configuration to deformed configuration is defined by the unknown y(x,t). The slope of the beam is symbolized by . When the beam deforms, the internal forces that occur in structures are bending moment (M) and shear force (V). In mathematics, the bending moment is proportion to curvature of the beams and shear force is derivative of bending moment. The equations for bending moment and shear force are presented below.

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{M}{EI} \qquad (1)$$

$$\frac{\partial^3 y(x,t)}{\partial x^3} = \frac{V}{EI} \qquad (2)$$

Formulation

The equation that describes the motion of beam can be presented as.

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + m\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \qquad (3)$$

where

E	= Young modulus
I	= Moment of inertia
m	= Mass per unit length
x	= Horizontal displacement
y(x,t)	= Deflection

To solve above equation, there are many methods to solve this equation. One is analytical method which closed form and exact solutions are obtained such as separate variable technique [1]. Another is numerical method which approximate solutions are obtained such as finite element method [2] and finite difference method [3,4]. However finite difference method is the simplest and high accuracy method under prescribe conditions. Then in this paper, we present the finite difference method for solving Eq (1).

Consider the first term of Eq (3), after discretizing by central difference which gives the accuracy in the order of $O\left(h^2, k^2\right)$. The result is.

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} = \frac{EI}{h^4}\left(y^n_{m+2} - 4y^n_{m+1} + 6y^n_m - 4y^n_{m-1} + y^n_{m-2}\right) \qquad (4)$$

Similarly for the second term of Eq (3).

$$m\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{m}{k^2}\left(y^{n+1}_m - 2y^n_m + y^{n-1}_m\right) \qquad (5)$$

Substituting the right hand side terms of Eqs (4) and (5) into Eq (3) we have.

$$\frac{EI}{h^4}\left(y^n_{m+2} - 4y^n_{m+1} + 6y^n_m - 4y^n_{m-1} + y^n_{m-2}\right) + \frac{m}{k^2}\left(y^{n+1}_m - 2y^n_m + y^{n-1}_m\right) = 0 \qquad (6)$$

k are h time and space intervals respectively.

Rearrange the Eq (6), the explicit numerical scheme is shown in the following form.

$$y^{n+1}_m = -a\left(y^n_{m+2} - 4y^n_{m+1} + 6y^n_m - 4y^n_{m-1} + y^n_{m-2}\right) + 2y^n_m - y^{n-1}_m \qquad (7)$$

where

$$a = \frac{EIk^2}{mh^4} \qquad (8)$$

As it can be seen in Eq (7), this is a three levels scheme. There is a difficulty to define the value at time step n-1. The procedure that can solve this problem is setting time step n-1 as initial time step, approximate deflection *y(x,t)* time step n by Taylor's series and *y(x,t)* at time step n+1 by Eq (7).

The deflection shape of the beams in time step n-1 (0) is known by initial configuration. This initial configuration may be called excited stage. The deflection *y(x,t)* at time step n (1) can be evaluated by expanding the Taylor's series about time step 0 as follows.

$$y_m^1 = y_m^0 + k(y_t)_m^0 + \frac{k^2}{2}(y_{tt})_m^0 + O(k^3) \qquad (9)$$

Let

$$f_m = y_m^0 \qquad (10)$$

$$g_m = (y_t)_m^0 \qquad (11)$$

From Eq (3),

$$y_m^1 = f_m + kg_m - \frac{a}{2}(f_{m+2} - 4f_{m+1} + 6f_m - 4f_{m-1} + f_{m-2}) \qquad (12)$$

Substituting Eqs (10)-(12) into Eq (9), we obtained

$$y_m^1 = f_m + kg_m - \frac{a}{2}(f_{m+2} - 4f_{m+1} + 6f_m - 4f_{m-1} + f_{m-2}) \qquad (13)$$

The Eqs (7) and (13) together with suitable boundary conditions are utilized in solving the solutions of the problem.

Consistency

To prove the consistency of Eq (7), one must expand deflection y(x,t) about m and n as follows.

$$y_m^{n+1} = y_m^n + k(y_t)_m^n + \frac{k^2}{2}(y_{tt})_m^n + \frac{k^3}{3!}(y_{ttt})_m^n + O(k^4) \qquad (14)$$

$$y_m^{n-1} = y_m^n - k(y_t)_m^n + \frac{k^2}{2}(y_{tt})_m^n - \frac{k^3}{3!}(y_{ttt})_m^n + O(k^4) \qquad (15)$$

$$y_{m+2}^n = y_m^n + 2h(y_x)_m^n + \left(\frac{4h^2}{2}\right)(y_{xx})_m^n + \left(\frac{8h^3}{3!}\right)(y_{xxx})_m^n + \left(\frac{16h^4}{4!}\right)(y_{xxxx})_m^n + \left(\frac{32h^5}{5!}\right)(y_{xxxxx})_m^n + O(h^6) \qquad (16)$$

$$y_{m-2}^n = y_m^n - 2h(y_x)_m^n + \left(\frac{4h^2}{2}\right)(y_{xx})_m^n - \left(\frac{8h^3}{3!}\right)(y_{xxx})_m^n + \left(\frac{16h^4}{4!}\right)(y_{xxxx})_m^n - \left(\frac{32h^5}{5!}\right)(y_{xxxxx})_m^n + O(h^6) \qquad (17)$$

$$y_{m+1}^n = y_m^n + h(y_x)_m^n + \left(\frac{h^2}{2}\right)(y_{xx})_m^n + \left(\frac{h^3}{3!}\right)(y_{xxx})_m^n + \left(\frac{h^4}{4!}\right)(y_{xxxx})_m^n + \left(\frac{h^5}{5!}\right)(y_{xxxxx})_m^n + O(h^6) \qquad (18)$$

$$y_{m-1}^n = y_m^n - h(y_x)_m^n + \left(\frac{h^2}{2}\right)(y_{xx})_m^n - \left(\frac{h^3}{3!}\right)(y_{xxx})_m^n + \left(\frac{h^4}{4!}\right)(y_{xxxx})_m^n - \left(\frac{h^5}{5!}\right)(y_{xxxxx})_m^n + O(h^6) \qquad (19)$$

Add the Eqs (14) with (15), Eqs (16) with (17) and Eqs (18) with (19) , we obtain the following equations respectively.

$$y_m^{n+1} + y_m^{n-1} = 2y_m^n + k^2(y_{tt})_m^n + O(k^4) \qquad (20)$$

$$y_{m+2}^n + y_{m-2}^n = 2y_m^n + 4h^2(y_{xx})_m^n + \frac{32h^4}{4!}(y_{xxxx})_m^n + O(h^6) \qquad (21)$$

$$y_{m+1}^n + y_{m-1}^n = 2y_m^n + h^2(y_{xx})_m^n + \frac{2h^4}{4!}(y_{xxxx})_m^n + O(h^6) \qquad (22)$$

Rearrange the Eq (17) into the following form.

$$y_m^{n+1} + y_m^{n-1} = -a\left(\left(y_{m+2}^n + y_{m-2}^n\right) - 4\left(y_{m+1}^n + y_{m-1}^n\right) + 6y_m^n\right) + 2y_m^n \qquad (23)$$

Substituting Eqs (20)-(22) into Eq (23), we have

$$2y_m^n + k^2(y_{tt})_m^n + O(k^4) = 2y_m^n - a\left(\begin{array}{l} 2y_m^n + 4h^2(y_{xx})_m^n + \frac{32h^4}{4!}(y_{xxxx})_m^n \\ - 4\left(2y_m^n + h^2(y_{xx})_m^n + \frac{2h^4}{4!}(y_{xxxx})_m^n\right) + 6y_m^n \\ + O(h^6) \end{array}\right) \qquad (24)$$

With some algebraic manipulation, we have

$$k^2(y_{tt})_m^n + O\left(k^4\right) = -a\left(h^4(y_{xxxx})_m^n + O(h^6)\right) \qquad (25)$$

Eq (25) is reduced to the following equation.

$$(y_{tt})_m^n + \frac{EI}{m}(y_{xxxx})_m^n + O\left(k^2 + h^2\right) = 0 \qquad (26)$$

Take the limit of the parameters h and k into zero. The Eq (26) becomes.

$$(y_{tt})_m^n + \frac{EI}{m}(y_{xxxx})_m^n = 0 \qquad (27)$$

Stability

The stability condition of this scheme is evaluated by using Von Neumann analysis. However the scheme in Eq (7) is three-level scheme, then the three-level scheme must be reduced into two-level scheme for using Von Neumann analysis. Let, then the Eq (7) can be rewritten into the system of equation as follows.

$$y_m^{n+1} = -a\left(y_{m+2}^n - 4y_{m+1}^n + 6y_m^n - 4y_{m-1}^n + y_{m-2}^n\right) + 2y_m^n - v_m^n \qquad (28)$$

$$v_m^{n+1} = y_m^n \qquad (29)$$

The Fourier transformation of $y_{m+1}^n, y_{m-1}^n, y_{m+2}^n$ and y_{m-2}^n are shown below.

$$y_{m+1}^n = e^{i\beta h} y_m^n = (\cos\beta h + i\sin\beta h)y_m^n \qquad (30)$$

$$y_{m-1}^n = e^{-i\beta h} y_m^n = (\cos\beta h - i\sin\beta h)y_m^n \qquad (31)$$

$$y_{m+2}^n = e^{i2\beta h} y_m^n = (\cos 2\beta h + i\sin 2\beta h)y_m^n \qquad (32)$$

$$y_{m-2}^n = e^{-i2\beta h} y_m^n = (\cos 2\beta h - i\sin 2\beta h)y_m^n \qquad (33)$$

Substituting Eqs (30)-(33) into Eq (28), the Eq (28) becomes.

$$y_m^{n+1} = \left(2 - a(2\cos 2\beta h - 8\cos\beta h + 6)\right)y_m^n - v_m^n \qquad (34)$$

The Eqs (34) and (29) can be rewritten in the matrix form as.

$$\begin{Bmatrix} y_m^{n+1} \\ v_m^{n+1} \end{Bmatrix} = \begin{bmatrix} 2 - a(2\cos 2\beta h - 8\cos\beta h + 6) & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} y_m^n \\ v_m^n \end{Bmatrix} \qquad (35)$$

By using the identity of trigonometry the Eq (35) transforms into Eq (36).

$$\begin{Bmatrix} y_m^{n+1} \\ v_m^{n+1} \end{Bmatrix} = \begin{bmatrix} 2 - 16a\sin^4\left(\frac{\beta h}{2}\right) & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} y_m^n \\ v_m^n \end{Bmatrix} \qquad (36)$$

Find the eigenvalue of coefficient matrix of Eq (36), the characteristic equation is obtained.

$$\lambda^2 - \left(2 - 16a\sin^4\left(\frac{\beta h}{2}\right)\right)\lambda + 1 = 0 \qquad (37)$$

The largest value of Eq (37) must be less than 1 to guarantee the stability of the solutions. The roots of Eq (37) are determined below.

$$\lambda_{1,2} = 1 - 8a\sin^4\left(\frac{\beta h}{2}\right) \pm 4\sqrt{a}\sin^2\left(\frac{\beta h}{2}\right)\sqrt{-1 + 4a\sin^4\left(\frac{\beta h}{2}\right)} \qquad (38)$$

In the case of, $-1 + 4a\sin^4\left(\frac{\beta h}{2}\right) < 0$ the Eq (38) is the complex number and shown in the following equation.

$$\lambda_{1,2} = 1 - 8a\sin^4\left(\frac{\beta h}{2}\right) \pm 4\sqrt{a}\sin^2\left(\frac{\beta h}{2}\right) i\sqrt{1 - 4a\sin^4\left(\frac{\beta h}{2}\right)} \qquad (39)$$

Where

$$i^2 = -1 \qquad (40)$$

The magnitudes of the Eq (39) are 1, and then we guarantee that under the condition of $-1 + 4a\sin^4\left(\frac{\beta h}{2}\right) < 0$ the solutions must converge.

From the stability condition, $-1 + 4a\sin^4\left(\frac{\beta h}{2}\right) < 0$ it is lead to

$$4a\sin^4\left(\frac{\beta h}{2}\right) < 1 \rightarrow a < \frac{1}{4} \qquad (41)$$

From Eq (41) the maximum value of $\sin^4\left(\frac{\beta h}{2}\right)$ is 1 then the maximum value of the left hand side is 4a. Hence $4a<1 \rightarrow a < \frac{1}{4}$ is stability condition that we guarantee.

Method of the solutions

The procedure of the solution for three major types of beam problem will discuss in this section. The Eqs (7) and (13) with the suitable boundary conditions play the crucial role in solving the problem. The picture below is the simple beam which will be discussed firstly.

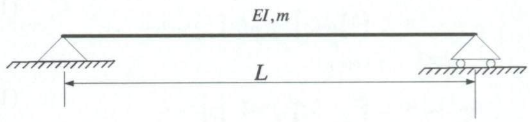

Fig. 3 Simple beam

From Fig. 3, the simple beam is discretized and shown in Fig. 4

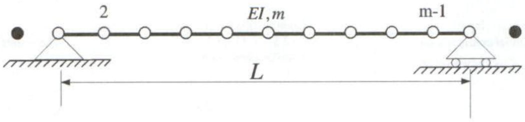

Fig. 4 Discretization of simple beam

The white points are the points that we must evaluate. Due to this scheme utilizes two nodes around the considered point. Then the difficulty can be arisen when node 2 and m-1 are considered. The black points do not know the values by initial configuration. However we can eliminate points which are out of domain by finding the relationship between black and white nodes. The black points are the points which out of domain (no need to evaluate the values) but need to find the relationship with the points with in domain (white points) for the solutions.

The relationship of black and white nodes can be found by boundary conditions. The boundary conditions of the simple beam is the bending moment and deflection at *x=0* and *x=L* are zero.

Boundary conditions:

$$\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=0} = 0 \underset{\text{discretization}}{\rightarrow} y_2^n - 2y_1^n + y_0^n = 0 \tag{42}$$

$$\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=L} = 0 \underset{\text{discretization}}{\rightarrow} y_{m+1}^n - 2y_m^n + y_{m-1}^n = 0 \tag{43}$$

$$\left. y(x,t)\right|_{x=0} = 0 \rightarrow y_1^n = 0 \tag{44}$$

$$\left. y(x,t)\right|_{x=L} = 0 \rightarrow y_m^n = 0 \tag{45}$$

Eqs (42)-(45) lead to the relationship in Eqs (46) and (47).

$$y_0^n = -y_2^n \tag{46}$$

$$y_{m+1}^n = -y_{m-1}^n \tag{47}$$

For the case of fixed-fixed beam, Figs. 5 and 6 are the fixed-fixed beam and discretized fixed-fixed beam respectively.

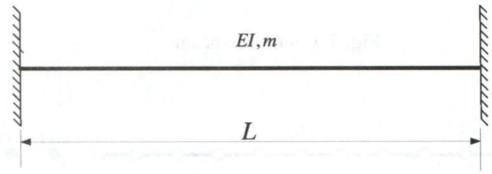

Fig. 5 Fixed-fixed beam

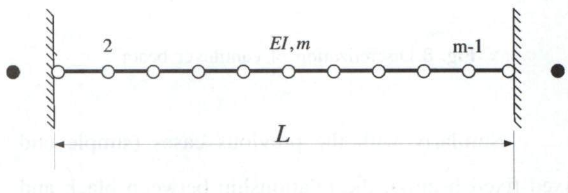

Fig. 6 Discretization of fixed-fixed beam

Similarly with the previous case (simple beam), the relationship between black and white nodes are found by boundary conditions. The boundary conditions for fixed-fixed beam are the slope and deflection at *x=0* and *x=L* are zero (see Eqs. (48)-(49)).

Boundary conditions:

$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = 0 \underset{\text{discretization}}{\rightarrow} y_2^n - y_0^n = 0 \tag{48}$$

$$\left.\frac{\partial y}{\partial x}\right|_{x=L} = 0 \underset{\text{discretization}}{\rightarrow} y_{m+1}^n - y_{m-1}^n = 0 \tag{49}$$

From Eqs (48) and (49), the relationships between black and white nodes are.

$$y_0^n = y_2^n \tag{50}$$

$$y_{m+1}^n = y_{m-1}^n \tag{51}$$

For the case of cantilever beam, Figs. 7 and 8 are cantilever in continuous and discretized models.

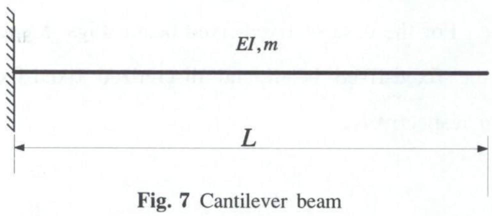

Fig. 7 Cantilever beam

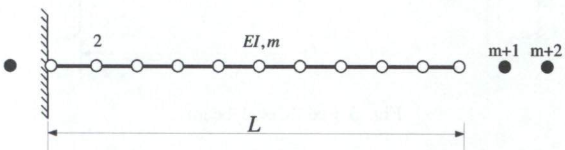

Fig. 8 Discretization of cantilever beam

Similarly with the previous cases (simple and fixed-fixed beams), the relationship between black and white nodes are found by boundary conditions. The boundary conditions for this case are *i)* the slope and deflection at *x=0* are zero *ii)* the bending moment and shear force at *x=L* are zero.

For the boundary conditions at *x=0*, the Eq (50) is obtained. At *x=L*, the equations of bending moment and shear force are described below.

$$\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=L} = \frac{M}{EI} = 0 \tag{52}$$

$$\left.\frac{\partial^3 y(x,t)}{\partial x^3}\right|_{x=L} = \frac{V}{EI} = 0 \tag{53}$$

After discretizing, the Eqs (54) and (55) are obtained.

$$y^n_{m+1} - 2y^n_m + y^n_{m-1} = 0 \tag{54}$$

$$y^n_{m+2} - 2y^n_{m+1} - 2y^n_{m-1} + y^n_{m-2} = 0 \tag{55}$$

The Eqs (54) and (55) are used for finding the value of.Below is the box diagram for computation the results.

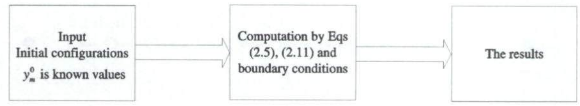

Fig. 9 Box diagram of computation.

From Fig. 9 the box diagram that shows the step of computation. The input value is initial configuration of the beams. In computation step, the finite difference scheme in Eqs (7), (13) and suitable boundary conditions are utilized. The results are displayed in the form of spatial configuration vary with time.

Natural Frequency

The differential equation which represents the behavior of frequency of the beams is expressed below.

$$\frac{d^4 y}{dx^4} - \lambda y = 0 \tag{56}$$

where

$$\lambda = \frac{m\omega^2}{EI} \tag{57}$$

The parameter that appears in Eq (57) is called natural frequency (rad/sec). Eq (56) is discretized into the following form.

$$y_{m+2} - 4y_{m+1} + 6y_m - 4y_{m-1} + y_{m-2} - \lambda h^4 y_m = 0 \tag{58}$$

Using Eq (58) with boundary conditions, we can construct the eigenvalue system in the form of Eq (59)

$$[A]\{v\} - b[I]\{v\} = \{0\} \rightarrow [A - bI]\{v\} = \{0\} \tag{59}$$

where

[A] is the coefficient matrix which depends on boundary conditions of the problem.

[I] is the identity matrix.

$b=\lambda h^4$ is the eigenvalue.

{v} is eigenvector.

For the case of simple beam, Eq (58) with boundary conditions in Eqs. (42) and (43) are utilized for obtaining the coefficient matrix.*[A]* In Fig. 4, we establish the system of equation by considering the nodes with in domain (white nodes). As previous mention, the nodes that locate out of the domain (black nodes) must be evaluated in terms of white nodes. Then the coefficient matrix is determined below.

$$[A]=\begin{bmatrix} 5 & -4 & 1 & 0 & 0 & 0 & 0 \\ 4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 1 & -4 & 6 & O & O & 0 & 0 \\ 0 & 1 & O & O & O & 1 & 0 \\ 0 & 0 & O & O & 6 & -4 & 1 \\ 0 & 0 & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & 1 & -4 & 5 \end{bmatrix} \qquad (60)$$

The matrix in Eq (60) has dimension by m-2xm-2 (m = number of spatial node). Then substitute Eq (60) into (59), we obtain.

$$\left\{\begin{bmatrix} 5 & -4 & 1 & 0 & 0 & 0 \\ 4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 1 & -4 & 5 \end{bmatrix} - \lambda h^4 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right\} \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{Bmatrix} = \{0\} \qquad (61)$$

Similarly for the case of fixed-fixed beam, the coefficient matrix is.

$$[A]=\begin{bmatrix} 7 & -4 & 1 & 0 & 0 & 0 & 0 \\ 4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 1 & -4 & 6 & O & O & 0 & 0 \\ 0 & 1 & O & O & O & 1 & 0 \\ 0 & 0 & O & O & 6 & -4 & 1 \\ 0 & 0 & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix} \qquad (62)$$

Eq (62) is the matrix of dimension. Substitute Eq (62) into (59), the result is.

$$\left\{\begin{bmatrix} 7 & -4 & 1 & 0 & 0 & 0 \\ 4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix} - \lambda h^4 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right\} \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{Bmatrix} = \{0\} \qquad (63)$$

For the case of cantilever beam, the coefficient matrix is.

$$[A]=\begin{bmatrix} 7 & -4 & 1 & 0 & 0 & 0 & 0 \\ 4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 1 & -4 & 6 & O & O & 0 & 0 \\ 0 & 1 & O & O & O & 1 & 0 \\ 0 & 0 & O & O & 6 & -4 & 1 \\ 0 & 0 & 0 & 1 & -4 & 5 & -2 \\ 0 & 0 & 0 & 0 & 2 & -4 & 2 \end{bmatrix} \qquad (64)$$

The Eq (64) is the matrix which has dimension equals *m-1 = m-1* . Substitute Eq (64) into (59), one obtains.

$$\left\{\begin{bmatrix} 7 & -4 & 1 & 0 & 0 & 0 \\ 4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -4 & 5 & -2 \\ 0 & 0 & 0 & 2 & -4 & 2 \end{bmatrix} - \lambda h^4 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right\} \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{Bmatrix} = \{0\} \qquad (65)$$

The Eqs (61), (63) and (65) are the equations that must be solved for the eigenvalues. As it can be seen from Eqs (61), (63) and (65) that there are many eigenvalues depend on size of matrix coefficient. However the natural or fundamental frequency is the lowest frequency corresponding with the lowest value of eigenvalues from the matrix.

Results and Discussion

The calculation of the results is performed by computer using FORTRAN code. Inputs of this program are *i)* Moment of inertia (*I*), *ii)* Young modulus (*E*), *iii)* Mass per unit length (*m*), *iv)* Number of node in space, *v)* Number of time step, and *vi)* Time interval. After input all of these parameters, the calculation will compute and return the results which are the position of points of each node. In this case, the input values are moment of inertia (I) = 1e-2, Young modulus (E) = 200, mass = 1, number of node for space = 10, number of time step = 300, time interval = 2e-3.

As it can be seen in Fig. 10, the beam is excited in the shape of sine wave (see circle symbol in Fig. 10. The beam also vibrates in the shape of sine wave. The frequency of vibration can be observed by tracking any node with time (see Fig.11). From observing, the period of the vibration is about 0.45 sec, then the frequency ($f = \frac{1}{T}$) is 2.2222 Hz. Normally the natural frequency is expressed in term of which shown below.

$$w = 2p\,f = \frac{2p}{T} \text{ (rad/sec)} \qquad (66)$$

Hence in this case ω =13.9625 rad/sec.

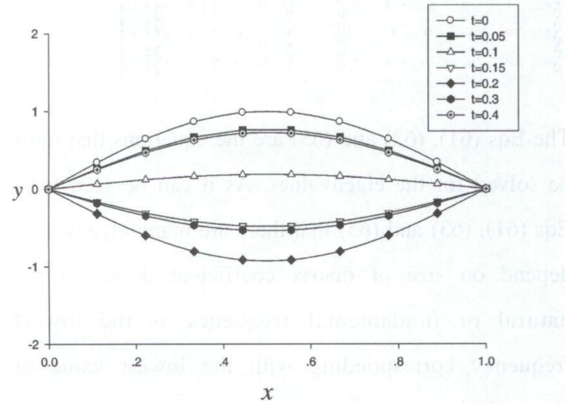

Fig. 10 Vibration of simple beam

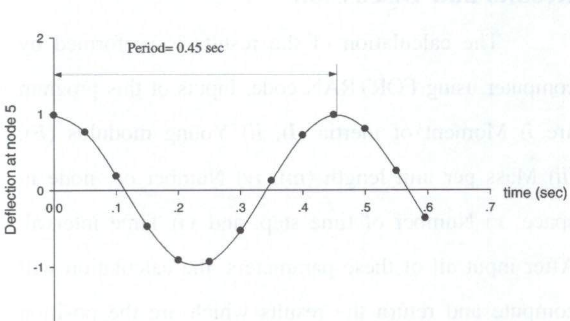

Fig. 11 Motion of node 5 of simple beam

To increase the accuracy of the results, the number of grid points should be increased. From Fig. 12, as the number of grid point increase (from 10 to 20 points) the results converge to the exact solutions. The equation that expresses the exact results is shown below.

$$y(x,t) = y(x,0)\cos(wt) \qquad (67)$$

where

$$y(x,0) = \sin\left(\frac{p x}{L}\right) \qquad (68)$$

$$w = p^2\sqrt{\frac{EI}{mL^4}} \qquad (69)$$

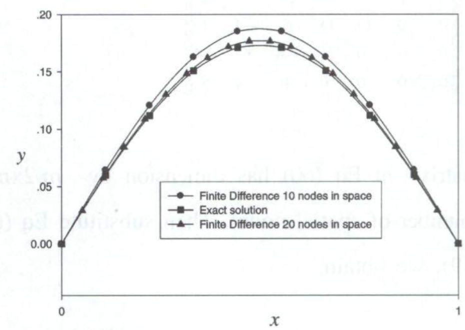

Fig.12 Numerical convergence of simple beam

The instability in numerical can occur if the criteria in Eq (41) is fail. The example of this event is presented in Fig. 13. The criterion for convergence is time interval must be less than 4.36485e-3 sec. If we set the time interval equal of 5e-3, the numerical instability is displayed in Fig. 13.

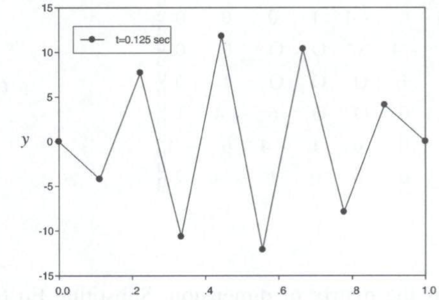

Fig. 13 Numerical unstable of simple beam

For the case of fixed-fixed beam, vibration of fixed-fixed beam is shown in Fig. 14. The input values are the same as the case of simple beam.

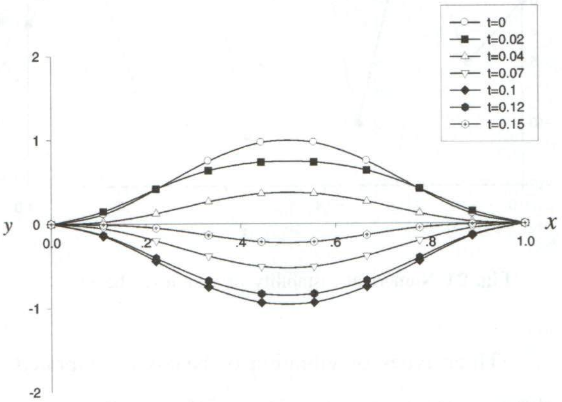

Fig. 14 Vibration of fixed-fixed beam

The results can be explained similarly as a previous case (simple beam). The beam is excited in to the sine wave. The vibration of this beam also vibrates in the shape of sine wave. The period (T) of this type of beam is observed and its value equal of 0.2 sec (see Fig. 15), then the frequency of this type of beam is 5 Hz or in unit of rad/sec is 31.4159 rad/sec (refer to Eq. (66)).

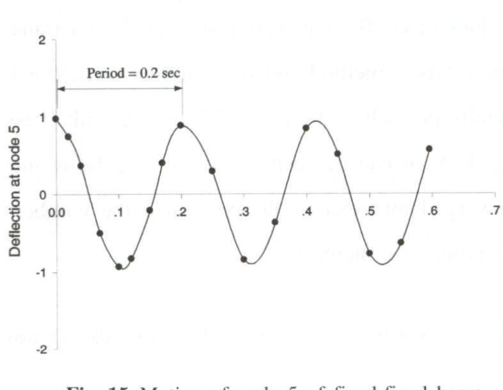

Fig. 15 Motion of node 5 of fixed-fixed beam

The accuracy of the result is increased by increasing the number of grid points. In this case, we increase the grid points from 10 to 20 points. Fig. 16 shows the convergence of the numerical results which closed to the exact solutions when grid points are increased.

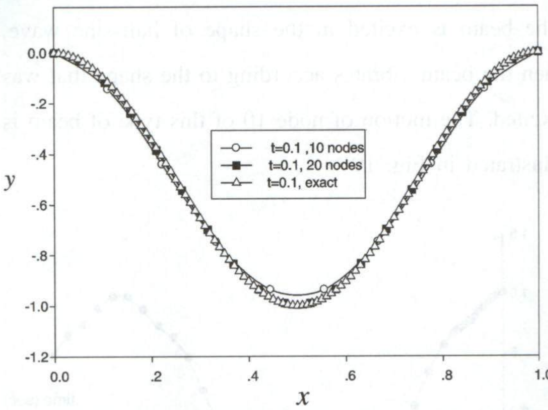

Fig. 16 Numerical convergence for fixed-fixed beam

The numerical instability when time interval equals of 5e-3 sec is shown in Fig. 17.

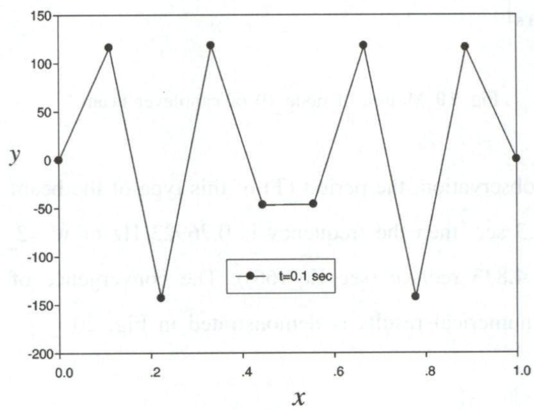

Fig. 17 Numerical instability of fixed-fixed beam

In the final case, cantilever beam, the vibration of cantilever beam is shown in Fig. 18. The input values are the same as two previous cases.

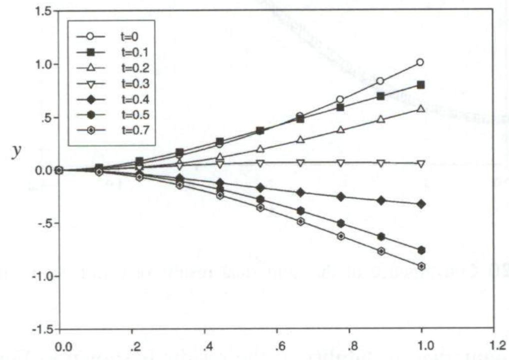

Fig. 18 Vibration of cantilever beam

The beam is excited in the shape of half-sine wave, then the beam vibrates according to the shape that was excited. The motion of node 10 of this type of beam is illustrated in Fig. 19

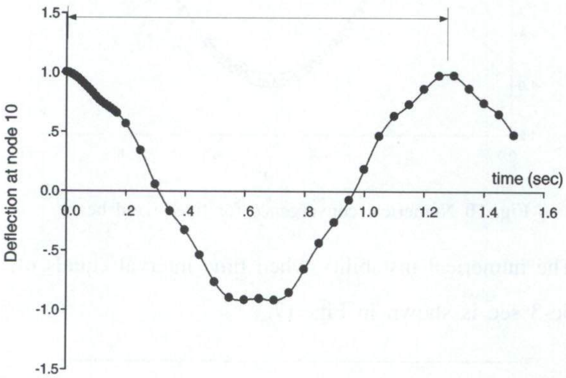

Fig. 19 Motion of node 10 of cantilever beam

By observation, the period (T) of this type of the beam is 1.3 sec, then the frequency is 0.76923 Hz or $w = 2_p f$ = 4.833 red/sec (see Eq (66)). The convergence of the numerical results is demonstrated in Fig. 20.

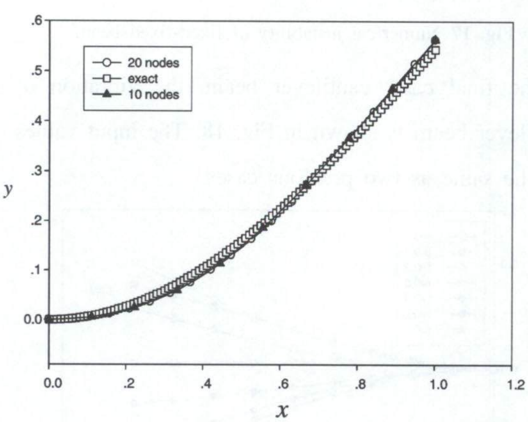

Fig. 20 Convergence of the numerical results of cantilever beam

The numerical instability of the results is shown in Fig. 21. The time interval of 5e-3 is the input of this case.

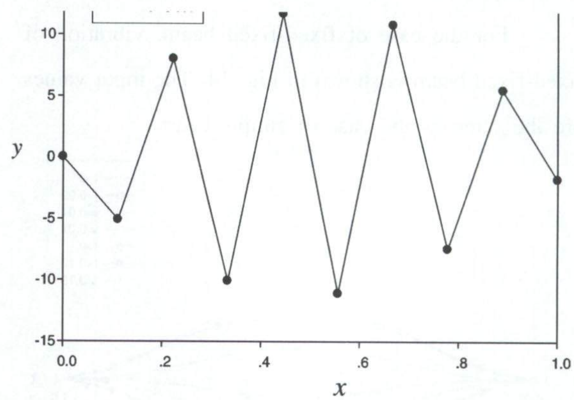

Fig. 21 Numerical instability of cantilever beam

Three types of vibration of beams are vibrated in difference frequency depend on stiffness *(EI/L)*, mass (*m*) of structures and boundary conditions. For example theses beams have same stiffness and mass but they vibrate with difference values of frequency then the boundary conditions play the important rule in natural frequency of the structures.

Natural frequency

The results of natural frequency are shown in Table 2. There are many numerical methods to find the eigenvalues of coefficient matrix such as Jacobi transformation, Given method and Householder method ect. The details of each method should consult with Press et. al. [5]. As it can be seen from Table 2, the results are converged into exact solutions when the numbers of grid points are increased.

Table 2. Comparison between the results of FDM and Exact solutions

Types of Beams	FDM		Exact solutions
	5 nodes	10 nodes	
simple beam	9.37059	9.92043	9.86960
fixed-fixed beam	17.9243	21.27696	22
cantilever beam	3.35619	3.48017	3.52

The values that appear in table are the values of coefficients of $\sqrt{\frac{EI}{mL^4}}$.

The values that appear in table are the values of coefficients of.

Conclusions

The finite difference scheme with order of accuracy $O(h^2 + k^2)$ gives the good agreement of the results. From the results, the conclusions may be drawn as follows.

1. The beam is excited with transcendental functions which correspond to the first or fundamental mode of vibration.
2. The frequency of vibration equals of natural frequency.
3. The results from finite difference schemes are good agreement with the exact solutions.
4. As increase the number of nodes of space the approximate solutions are converged to the exact solutions.

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