Jump-Diffusion with Stochastic Volatility and Intensity

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Abstract: An alternative option pricing model is proposed, in which the asset prices follow the jump-diffusion with stochastic volatility and intensity. The stochastic volatility follows the jump-diffusion. We find a formulation for the European-style option in terms of characteristic functions.

Keywords: Jump-diffusion model, Stochastic Volatility, Intensity, Characteristic functions.

1. Introduction

In 1973, Fischer Black and Myron Scholes introduced, a theoretical valuation formula for options is derived. In 1993, Heston studied a new technique to derive a closed – form solution for the price of a European call option on an asset with stochastic volatility. The Heston model assumes that S_t, the price of the asset, is determined by a stochastic process:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^S \tag{1}$$

where $\mu > 0$, v_t the instantaneous variance is a CIR process:

$$dv_t = \kappa_v(\theta_v - v_t)dt + \sigma\sqrt{v_t} dW_t^v \tag{2}$$

and $\kappa_v > 0, \theta_v > 0, \sigma > 0$, W_t^S, W_t^v are Brownian motion with correlation ρ.

In 1996, Bates introduced an efficient method is developed for pricing American options on stochastic volatility /jump-diffusion processes under systematic jump and volatility risk. The exchange rate S_t satisfy the following process:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^S + kdN_t \\ dv_t &= \kappa_v(\theta_v - v_t)dt + \sigma\sqrt{v_t} dW_t^v \end{aligned} \tag{3}$$

where k is the random percentage jump conditional on a jump occurring and N_t is a Poisson process with constant intensity λ.

2. Model Descriptions

The propose model assumes that the underlying asset has the following dynamics under risk-neutral measure,

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-\lambda_t m)dt + \sqrt{v_t}\,dW_t^S + Y_t dN_t \\ dv_t &= \kappa_v(\theta_v - v_t)dt + \sigma\sqrt{v_t}\,dW_t^v \\ d\lambda_t &= \kappa_\lambda(\theta_\lambda - \lambda_t)dt + \varepsilon\sqrt{v_t}\,dW_t^\lambda \end{aligned} \tag{4}$$

where S_t, v_t, κ_v, θ_v, σ, Y_t, N_t, W_t^S and W_t^v are define (1), (2) and (3). r is the risk-free rate , m is the expected of Y_t, κ_λ is a mean-reverting rate. We assume that jump process N_t are independent of W_t^S, W_t^v and W_t^λ. A standard Brownian motion W_t^λ, W_t^S and W_t^v are independent.

3. Characteristic Functions

Denote the characteristic function as

$$f(l,v,\lambda,t;x) = E[e^{ixX_T} \mid X_t = l, v_t = v] \tag{5}$$

where $T \ge t$ and $i = \sqrt{-1}$. Then, the following theorem holds.

Theorem 3.1 Suppose that S_t follows the dynamics in (4). Then the characteristic function for X_T defined in (5) is given by

$$f(l,v,\lambda,t;x) = \exp(ixl + ixr\tau + A(\tau) + B(\tau)v + C(\tau)\lambda),$$

where $A(\tau) = -\dfrac{2\kappa_v\theta_v}{\sigma^2}\ln\left[\dfrac{r_2 e^{-\frac{1}{2}r_1\tau} + r_1 e^{\frac{1}{2}r_2\tau}}{2H}\right] - \dfrac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\left[\dfrac{q_2 e^{-\frac{1}{2}q_1\tau} + q_1 e^{\frac{1}{2}q_2\tau}}{2E}\right]$,

$$B(\tau) = (u^2-u)\left(\frac{1-e^{-H\tau}}{r_1 + r_2 e^{-H\tau}}\right), \quad C(\tau) = 2F\left[\frac{1-e^{-E\tau}}{q_1 + q_2 e^{-E\tau}}\right], \quad u = ix$$

$$r_1 = (\kappa_v - \rho\sigma u) + H, \quad r_2 = -(\kappa_v - \rho\sigma u) + H, \quad H = \sqrt{(\kappa_v - \rho\sigma u)^2 - \sigma^2(u^2 - u)}$$

$$q_1 = \kappa_\lambda + E, \quad q_2 = -\kappa_\lambda + E, \quad E = \sqrt{\kappa_\lambda^2 - 2\varepsilon^2 F}, \quad F = -mu + \int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy$$

and $\phi_Y(y)$ is a density of random jump size Y_t.

Proof Feynman-Kac formula gives the following PDE for the characteristic function

$$(r - \frac{1}{2}v - \lambda m)f_l + \frac{1}{2}vf_{ll} + \kappa_v(\theta_v - v)f_v + \frac{1}{2}\sigma^2 v f_{vv} + \rho\sigma v f_{lv} + \kappa_\lambda(\theta_\lambda - v)f_\lambda$$

$$+\frac{1}{2}\varepsilon^2\lambda f_{\lambda\lambda} + \lambda\int_{-\infty}^{\infty}[f(l+y,v,\lambda,t;\phi) - f(l,v,\lambda,t;\phi)]\phi_Y(y)dy + f_t = 0, \tag{6}$$

$f(l,v,\lambda,T;x)=e^{ixl}$.

Consider form for the characteristic function:

$$f(l,v,\lambda,t;x)=\exp(ixl+ixr\tau+A(\tau)+B(\tau)v+C(\tau)\lambda) \tag{7}$$

where $\tau=T-t$ and $A(\tau=0)=B(\tau=0)=C(\tau=0)$.

We plan to substitute equation (7) into equation (6). Firstly, we compute

$$f_l=ixf,\ f_{ll}=-x^2f,\ f_v=B(\tau)f,\ f_{vv}=B^2(\tau)f,\ f_{lv}=ixB(\tau)f,\ f_\lambda=C(\tau)f,$$
$$f_{\lambda\lambda}=C^2(\tau)f,\ f_t=(-ixr-A_\tau-B_\tau v-C_\tau\lambda)f,$$
$$f(l+y,v,\lambda,t;x)-f(l,v,\lambda,t;x)=e^{ixy}f.$$

Substitute all terms above in equation (6),

$$(r-\frac{1}{2}v-\lambda m)ixf+\frac{1}{2}v(-x^2f)+\kappa_v(\theta_v-v)B(\tau)f+\frac{1}{2}\sigma^2vB^2(\tau)f+\rho\sigma vixB(\tau)f$$
$$+\kappa_\lambda(\theta_\lambda-\lambda)C(\tau)f+\frac{1}{2}\varepsilon^2\lambda C^2(\tau)f+\lambda f\int_{-\infty}^{\infty}e^{ixy}\phi_Y(y)dy-(ixr+A_\tau+B_\tau v+C_\tau\lambda)f=0.$$

Let $ix=u$, then

$$(r-\frac{1}{2}v-\lambda m)u+\frac{1}{2}vu^2+\kappa_v(\theta_v-v)B(\tau)+\frac{1}{2}\sigma^2vB^2(\tau)+\rho\sigma vuB(\tau)$$
$$+\kappa_\lambda(\theta_\lambda-\lambda)C(\tau)+\frac{1}{2}\varepsilon^2\lambda C^2(\tau)+\lambda\int_{-\infty}^{\infty}e^{uy}\phi_Y(y)dy-ru-A_\tau-B_\tau v-C_\tau\lambda=0.$$

We have

$$\begin{aligned}A_\tau+B_\tau v+C_\tau\lambda&=\kappa_v\theta_vB(\tau)+\kappa_\lambda\theta_\lambda C(\tau)\\&+\left(\frac{1}{2}u^2-\frac{1}{2}u-\kappa_vB(\tau)+\frac{1}{2}\sigma^2B^2(\tau)+\rho\sigma uB(\tau)\right)v\\&+\left(\frac{1}{2}\varepsilon^2C^2(\tau)-\kappa_\lambda C(\tau)-mu+\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy\right)\lambda.\end{aligned}$$

This leads to the following system :

$$A_\tau=\kappa_v\theta_vB(\tau)+\kappa_\lambda\theta_\lambda C(\tau) \tag{8}$$

$$B_\tau=-\frac{1}{2}(u-u^2)-(\kappa_v-\rho\sigma u)B(\tau)+\frac{1}{2}\sigma^2B^2(\tau) \tag{9}$$

$$C_\tau=\frac{1}{2}\varepsilon^2C^2(\tau)-\kappa_\lambda C(\tau)-mu+\int_{-\infty}^{\infty}(e^{uy}-1)\phi_Y(y)dy. \tag{10}$$

In the equation (9) become a Ricatti equation. Let

$$B(\tau)=-\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)},$$

substitute $B(\tau)$ in equation (9),

$$-\left[\frac{\sigma^2}{2}G(\tau)G''(\tau)-\frac{\sigma^2}{2}(G'(\tau))^2\right]\frac{1}{\frac{\sigma^4}{4}G^2(\tau)}=-\frac{1}{2}(u-u^2)+(\kappa_v-\rho\sigma u)\frac{G'(\tau)}{\frac{\sigma^2}{2}G^2(\tau)}+\frac{\frac{1}{2}\sigma^2(G'(\tau))^2}{\frac{\sigma^4}{4}G^2(\tau)}$$

Then

$$\frac{\sigma^2}{2}\frac{G(\tau)G''(\tau)}{\frac{\sigma^4}{4}G^2(\tau)}+\frac{1}{2}(u^2-u)-(\kappa_v-\rho\sigma u)\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}=0\,.$$

Multiply by $\frac{\sigma^2}{2}G(\tau)$,

$$G''(\tau)+(\kappa_v-\rho\sigma u)G'(\tau)+\frac{\sigma^2}{4}(u^2-u)G(\tau)=0\,.$$

General solution is

$$\begin{aligned}G(\tau)&=C_1e^{\frac{-(\kappa_v-\rho\sigma u)-\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}}{2}\tau}+C_2e^{\frac{-(\kappa_v-\rho\sigma u)+\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}}{2}\tau}\\&=C_1e^{\frac{-1}{2}r_1\tau}+C_2e^{\frac{1}{2}r_2\tau}\end{aligned}$$

where

$$r_1=(\kappa_v-\rho\sigma u)+H\,,\ H=\sqrt{(\kappa_v-\rho\sigma u)^2-\sigma^2(u^2-u)}$$

$$r_2=-(\kappa_v-\rho\sigma u)+H\,.$$

Note that $r_1+r_2=2H,\ r_1r_2=-\sigma^2(u^2-u)$.

The boundary condition

$$G(0)=C_1+C_2$$

$$G'(0)=\frac{-1}{2}r_1C_1+\frac{1}{2}r_2C_2=0\,.$$

We have $C_1=\frac{r_2G(0)}{2H}$ and $C_2=\frac{r_1G(0)}{2H}$.

Thus

$$B(\tau)=-\frac{G'(\tau)}{\frac{\sigma^2}{2}G(\tau)}=\frac{-\frac{1}{2}r_1\frac{r_2G(0)}{2H}e^{-\frac{1}{2}r_1\tau}+\frac{1}{2}r_2\frac{r_1G(0)}{2H}e^{\frac{1}{2}r_2\tau}}{-\frac{\sigma^2}{2}[\frac{r_2G(0)}{2H}e^{-\frac{1}{2}r_1\tau}+\frac{r_1G(0)}{2H}e^{\frac{1}{2}r_1\tau}]}$$

$$=\frac{1}{\sigma^2}\left[\frac{r_1r_2e^{-\frac{1}{2}r_1\tau}-r_1r_2e^{\frac{1}{2}r_2\tau}}{r_2e^{-\frac{1}{2}r_1\tau}+r_1e^{\frac{1}{2}r_2\tau}}\right]$$

$$=\frac{1}{\sigma^2}\left[\frac{-\sigma^2(u^2-u)e^{-\frac{1}{2}r_1\tau}+\sigma^2(u^2-u)e^{\frac{1}{2}r_2\tau}}{r_2e^{-\frac{1}{2}r_1\tau}+r_1e^{\frac{1}{2}r_2\tau}}\right]$$

$$= (u^2 - u)\left[\frac{-e^{-\frac{1}{2}r_1\tau} + e^{\frac{1}{2}r_1\tau}}{r_2 e^{-\frac{1}{2}r_1\tau} + r_1 e^{\frac{1}{2}r_2\tau}}\right]$$

$$= (u^2 - u)\left(\frac{1 - e^{-H\tau}}{r_1 + r_2 e^{-H\tau}}\right).$$

Next, consider in equation (10).

$$C_\tau = \frac{1}{2}\varepsilon^2 C^2(\tau) - \kappa_\lambda C(\tau) - mu + \int_{-\infty}^{\infty} (e^{uy} - 1)\phi_Y(y)dy\,.$$

Let

$$C(\tau) = -\frac{M'(\tau)}{\frac{\varepsilon^2}{2}M(\tau)}\,.$$

Similarly in $B(\tau)$, we have

$$M(\tau) = \frac{q_2 M(0)}{2E} e^{-\frac{1}{2}q_1\tau} + \frac{q_1 M(0)}{2E} e^{\frac{1}{2}q_2\tau}$$

where $E = \sqrt{\kappa_\lambda^2 - 2\varepsilon^2 F}$, $F = -mu + \int_{-\infty}^{\infty} (e^{uy} - 1)\phi_Y(y)dy$, $q_1 = \kappa_\lambda + E$, $q_2 = -\kappa_\lambda + E$.

Thus

$$C(\tau) = \frac{-\frac{1}{2}q_1 \frac{q_2 M(0)}{2E} e^{-\frac{1}{2}q_1\tau} + \frac{1}{2}q_2 \frac{q_1 M(0)}{2E} e^{\frac{1}{2}q_2\tau}}{-\frac{\varepsilon^2}{2}[\frac{q_2 M(0)}{2E} e^{-\frac{1}{2}q_1\tau} + \frac{q_1 M(0)}{2E} e^{\frac{1}{2}q_2\tau}]}$$

$$= \frac{q_1 q_2 e^{-\frac{1}{2}q_1\tau} - q_1 q_2 e^{\frac{1}{2}q_2\tau}}{\varepsilon^2 (q_2 e^{-\frac{1}{2}q_1\tau} + q_1 e^{\frac{1}{2}q_2\tau})}$$

$$= \frac{1}{\varepsilon^2}(2\varepsilon^2 F)\left[\frac{1 - e^{-E\tau}}{q_1 + q_2 e^{-E\tau}}\right]$$

$$= 2F\left[\frac{1 - e^{-E\tau}}{q_1 + q_2 e^{-E\tau}}\right].$$

Consider in equation (8),

$$A_\tau = \kappa_v \theta_v B(\tau) + \kappa_\lambda \theta_\lambda C(\tau)\,.$$

Integrating with respect to τ ,

$$A(\tau) = \kappa_v \theta_v \int_0^\tau B(s)ds + \kappa_\lambda \theta_\lambda \int_0^\tau C(s)ds$$

$$= -\frac{2\kappa_v \theta_v}{\sigma^2}\int_0^\tau \frac{G'(s)}{G(s)}ds - \frac{2\kappa_\lambda \theta_\lambda}{\varepsilon^2}\int_0^\tau \frac{M'(s)}{M(s)}ds$$

$$= -\frac{2\kappa_v\theta_v}{\sigma^2}\ln G(s)\,|_{s=0}^{\tau} - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln M(s)\,|_{s=0}^{\tau}$$

$$= -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\frac{G(\tau)}{G(0)} - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\frac{M(\tau)}{M(0)}$$

$$= -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\left[\frac{r_2G(0)e^{-\frac{1}{2}r_1\tau}}{2HG(0)} + \frac{r_1G(0)e^{\frac{1}{2}r_2\tau}}{2HG(0)}\right] - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\left[\frac{q_2M(0)e^{-\frac{1}{2}q_1\tau}}{2EM(0)} + \frac{q_1M(0)e^{\frac{1}{2}q_2\tau}}{2EM(0)}\right]$$

$$A(\tau) = -\frac{2\kappa_v\theta_v}{\sigma^2}\ln\left[\frac{r_2e^{-\frac{1}{2}r_1\tau} + r_1e^{\frac{1}{2}r_2\tau}}{2H}\right] - \frac{2\kappa_\lambda\theta_\lambda}{\varepsilon^2}\ln\left[\frac{q_2e^{-\frac{1}{2}q_1\tau} + q_1e^{\frac{1}{2}q_2\tau}}{2E}\right].$$

The proof is now completed.

4. A Formula for European Option Pricing

Following Carr and Madan (1999) , the modified call price $c_T(k)$ is defined by

$$c_T(k) = e^{\alpha k}C_T(k) \qquad \text{for some constant } \alpha > 0$$

where $C_T(k) = \int_k^{\infty} e^{-rT}(e^s - e^k)q_T(s)ds$ is the value of a T maturity call option with strike price e^k ($k = \ln K$), and $q_T(s)$ be the risk-neutral density of the log asset price $s_T = \ln S_T$. As $C_T(k)$ is not square integrable over $(-\infty,\infty)$, the introduction of a damping factor $e^{\alpha k}$ aims at removing this problem.

Theorems 3.2 The Fourier transform of $c_T(k)$ exist:

$$\psi_T(\xi) = \int_{-\infty}^{\infty} e^{i\xi k}c_T(k)\,dk$$

Proof

$$\psi_T(\xi) = \int_{-\infty}^{\infty} e^{i\xi k}\int_k^{\infty} e^{\alpha k}e^{-r_T}(e^s - e^k)q_T(s)dsdk$$

$$= \int_{-\infty}^{\infty} e^{-r_T}q_T(s)\int_{-\infty}^{s}(e^{\frac{(\alpha+1+i\xi)s}{\alpha+i\xi}} - e^{\frac{(\alpha+1+i\xi)s}{\alpha+1+i\xi}})ds$$

$$= \frac{e^{-rT}f(l,v,\lambda,t;x = \xi-(\alpha+1)i)}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}, \qquad (11)$$

where f is the characteristic function defined in theorem 3.1

A sufficient condition for c_T to be square-intefrable is given by $\psi_T(0)$ being finite. This is equivalent to

$$E(S_T^{\alpha+1}) < \infty.$$

Call prices can then be numerically obtained by using the inverse transform:

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi k} \psi_T(\xi)\, d\xi$$

$$= \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-i\xi k} \psi_T(\xi)\, d\xi \qquad (12)$$

More precisely, the call price is determined by substituting (11) into (12) and performing the required integration. Integration (12) is a direct Fourier transform and lends itself to an application of the FFT.

References

[1] F., Black, M., Scholes, "The Pricing of Options and Corporate liabilities", Journal of Political Economy 81, (1973), 637-654.

[2] D., Bates, "Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options", Review of Financial Studies (1996), 69-107.

[3] S.L.,Heston, "A Closed-Form Solution for Options with Stochastic Volatility with Application to Bond and Currency Options", The Review of Financial Studies 6 (2), (1993), 327-343.

[4] T. Mikosch, Elementary Stochastic Calaculus. (1998).

[5] P.Carr, D.Madan Option pricing and the Fast Fouries transform, Journal of Comptational Finance 4 (1999) 61-73.