On the Diophantine Equation $p^x + p^y = z^2$

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Abstract

In this paper, we study the diophantine equation $p^x + p^y = z^2$ where p is a prime number and x, y and z are non-negative integers.

2010 Mathematics Subject Classification: 11D61.

Key words and phrases: diophantine equations, exponential equations, Catalan's conjecture

1 Introduction

In 2007 Acu [1] studied the diophantine equation of form $2^x + 5^y = z^2$. He found that this equation has exactly two solutions in non-negative integer $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$. After that Suvarnamani, Singta and Chotchaisthit [7] found solutions of two diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. Then Suvarnamani [6] studied the diophantine equation of form $2^x + p^y = z^2$ where p is a prime number and x, y and z are non-negative integers.

Now, we study the diophantine equation $p^x + p^y = z^2$ where p is prime number and x, y and z are non-negative integers.

2 Main Results

In this study, we use Catalan's conjecture (see [4]). It is proved there that the only solution in integers $a > 1$, $b > 1$, $x > 1$ and $y > 1$ of the equation $a^x + b^y = 1$ is $a = y = 3$ and $b = x = 2$. From the diophantine equation

$$p^x + p^y = z^2 \qquad (1),$$

we consider in 5 cases.

Case 1: $x = 0$. The diophantine equation (1) becomes $(z-1)(z+1) = p^y$. Then we have $p^u(p^{y-2u} - 1) = 2$, where $z - 1 = p^u$ and $z + 1 = p^{y-u}$, for $y > 2u$ and u is a non-negative integer. Therefore, $p^u = 1$ or $p^u = 2$.
If $p^u = 1$, then $u = 0$. It follows that the diophantine equation (1) has a solution if $p = 3$,i.e., $x = 0, y = 1$ and $z = 2$.
If $p^u = 2$, then $p = 2$ and $u = 1$. It follows that the diophantine equation (1) has a solution ,i.e., $x = 0, y = 3$ and $z = 3$.

Case 2: $y = 0$. The diophantine equation (1) becomes $(z-1)(z+1) = p^x$. Then we have $p^v(p^{x-2v} - 1) = 2$, where $z - 1 = p^v$ and $z + 1 = p^{y-v}$, for $y > 2v$ and v is a non-negative integer. Therefore, $p^v = 1$ or $p^v = 2$.
If $p^v = 1$, then $v = 0$. It follows that the diophantine equation (1) has a solution if $p = 3$,i.e., $x = 1, y = 0$ and $z = 2$.
If $p^v = 2$, then $p = 2$ and $v = 1$. It follows that the diophantine equation (1) has a solution ,i.e., $x = 3, y = 0$ and $z = 3$.

Case 3: $x \geq 1$, $y \geq 1$ and $x = y$. We have $2p^x = z^2$. Then $p = 2$. We get $2^{x+1} = z^2$. It follows that $z = 2^k$ and $x = 2k - 1$ where $k \in N$.

Case 4: $x \geq 1$, $y \geq 1$ and $x > y$. We have $p^y(p^{x-y} + 1) = z^2$. Then $y = 2k$ and $p^{x-2k} + 1 = w^2$ where k and w are positive integers such that $z = p^k w$. We get $p^{x-2k} = w^2 - 1$. Now, it is easy to see that the solution of the diophantine equation (1) is $(x, y, z) \in \{(2k+1, 2k, 2 \cdot 3^k), k \in N\}$ for $p = 3$. And $(x, y, z) \in \{(2k+3, 2k, 3 \cdot 2^k),\ k \in N\ \}$ is a solution of the diophantine equation (1) where $p = 2$.

Case 5: $x \geq 1$, $y \geq 1$ and $x < y$. We get the solution of the diophantine equation (1) is $(x, y, z) \in \{(2k, 2k+1, 2 \cdot 3^k), k \in N\}$ for $p = 3$. And $(x, y, z) \in \{(2k, 2k+3, 3 \cdot 2^k),\ k \in N\ \}$ is a solution of the diophantine equation (1) where $p = 2$.

In conclusion, we have

Theorem 2.1. *Consider the diophantine equation (1) where p is a prime number,*

(i) *For* $p = 2$*, a solution of the diophantine equation (1) is* $(x, y, z) \in \{(0,3,3),(3,0,3)\} \cup \{(2k-1, 2k-1, 2^k) | k \in N\} \cup \{(x,y,z) \in \{(2k+3, 2k, 3 \cdot 2^k), k \in N\} \cup \{(x,y,z) \in \{(2k, 2k+3, 3 \cdot 2^k), k \in N\}$.

(ii) *For* $p = 3$*, a solution of the diophantine equation (1) is* $(x, y, z) \in (0,1,2),(1,0,2) \cup \{(x,y,z) \in \{(2k+1, 2k, 2 \cdot 3^k), k \in N\} \cup \{(x,y,z) \in \{(2k, 2k+1, 2 \cdot 3^k), k \in N\}$.

Acknowledgements

This research was partly supported by Faculty of Sciences and Technology, RMUTT, Thailand.

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