PSEUDO PRINCIPALLY QUASI-INJECTIVE MODULES AND PSEUDO QUASI-PRINCIPALLY INJECTIVE MODULES

NONTHIT GOONWISES

A THESIS SUBMITTED IN PARTIAL FULLFILLMENT OF THE

REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE

PROGRAM IN MATHEMATICS FACULTY OF SCIENCE AND TECHNOLOGY

RAJAMANGALA UNIVERSITY OF TECHNOLOGY THANYABURI

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Thesis Title	Pseudo Principally Quasi-injective Modules and Pseudo Quasi -Principally injective Modules
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Program	Mathematics
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Academic Year	2012

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ABSTRACT

The purposes of this thesis are to (1) study the properties and characterizations of pseudo principally quasi-injective modules and pseudo quasi-principally injective modules, (2) study the properties and characterizations of endomorphism rings of the two types of modules, (3) extend the concepts of principally quasi-injective modules and quasi-principally injective modules and (4) find some relations between among the four types of modules mentioned.

Let R be a ring. A right R-module M is called *principally injective* if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. A right R-module N is called *principally M-injective* if every R-homomorphism from a principal submodule of M to N can be extended to an R-homomorphism from M to N. A right R-module M is called *principally quasi-injective* if it is principally M-injective. A right R-module N is called *M-principally injective* if every R-homomorphism from an M-cyclic submodule of M to N can be extended to an R-homomorphism from M to N. A right R-module M is called *quasi-principally injective* if it is M-principally injective. The notion of principally quasi-injective modules and quasi-principally injective modules are extended to be pseudo principally quasi-injective modules and pseudo quasi-principally injective modules, respectively. A right R-module N is called *pseudo principally M-injective* if every R-monomorphism from a principal submodule of M to N can be extended to an R-homomorphism from M to N. A right R-module M is called *pseudo principally qausi-injective* if it is pseudo principally M-injective. A right R-module N is called *pseudo M-principally injective* if every R-monomorphism from an M-cyclic submodule of M to N

can be extended to an R-homomorphism from M to N. A right R-module M is called *pseudo qausi-principally injective* if it is pseudo M-principally injective.

The results are as follows. (1) Let M be a principal and pseudo principally quasi-injective module: (a) if M is weakly co-Hopfian, then M is co-Hopfian; (b) for a fully invariant essential submodule X of M, if X is weakly co-Hopfian, then M is weakly co-Hopfian; (c) if X is a principal and essential submodule of M and M is weakly co-Hopfian, then X is weakly co-Hopfian. (2) Let M be a pseudo quasi-principally injective module and $s, t \in S = End_R(M)$: (a) if $s(M)$ embeds in $t(M)$, then Ss is an image of St; (b) if $s(M) \cong t(M)$, then $Ss \cong St$. (3) Let M be a pseudo quasi-principally injective module and $S = End_R(M)$: (a) if $S/W(S)$ is regular, then $J(S) = W(S)$; (b) if $S/J(S)$ is regular, then $S/W(S)$ is regular if and only if $J(S) = W(S)$; (c) if $Im(s) \subset^e M$ where $s \in S$, then any R-monomorphism $\varphi : s(M) \to M$ can be extended to an R-monomorphism in S. (4) For a pseudo quasi-principally injective module M, if S is semiregular, then for every $s \in S \setminus J(S)$, there exists a nonzero idempotent $\alpha \in Ss$ such that $Ker(s) \subset Ker(\alpha)$ and $Ker(s(1-\alpha)) \neq 0$.

Keywords: pseudo principally quasi-injective modules, principally quasi-injective modules, endomorphism rings

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Nonthit Goonwises

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List of Abbreviations

$A \oplus B$	A direct sum B
$End_R(M)$	The set of R-homomorphism from M to M
F	Field F
$f : M \to N$	A function f from M to N
$f(M)$, $Im(f)$	Image of f
$Hom_R(M,N)$	The set of R-homomorphism from M to N
$Ker(f)$	Kernel of f
$J(M)$, $Rad(M_R)$	Jacobson radical of a right R-module M
$J(R) = Rad(R_R)$	Jacobson radical of a ring R
$J(S)$	Jacobson radical of a ring S
$J(S) \subset {}_S S_S$	$J(S)$ is an (two-side) ideal of ring S
$l_M(A)$	Left annihilator of A in M
M_R	M is a right R-module
$M_1 \times M_2$	Cartesian products of M_1 and M_2
M/K	A factor module of M modulo K or a factor module of M by K
$M \cong N$	M isomorphic N
R	Ring R
R_R	Ring R is a right R-module is called Regular right R-module
$r_R(X)$	Right annihilator of X in R
$Z(M)$	Singular submodule of M
1_M	Identity map on a module M
$\begin{pmatrix} F & F \\ F & F \end{pmatrix} = M_2(F)$	The set of all 2×2 matrices having elements of a field F as entries

List of Abbreviations (Continued)

$\eta : M \to M/K$	η (*eta*) is the natural epimorphism of M onto M/K
$\iota = \iota_{A \subseteq B} : A \to B$	ι (*iota*) is the inclusion map of A in B
π_j	π_j is the j-th projection map
$\forall$	For all
$\cap$	Intersection of set
$\not\subset$	is not subset
$\subset$	subset
$\in$	is in, member of set
$\subset^e$	Essential (Large) submodule
$\ll$	Superfluous (Small) submodule
$\prod_{i \in I} N_i$	Direct product of N_i
$\bigoplus_{i=1}^{n} N_i$	Direct sum of N_i

CHAPTER 1

INTRODUCTION

In modules and rings theory research field, there are three methods for doing the research. Firstly, to study about the fundamental of algebra and modules theory over arbitrary rings. Secondly, to study about the modules over special rings. Thirdly, to study about ring R by way of the categories of R-modules. Many mathematicians have concentrated on these methods.

1.1 Background and Statement of the Problems

Many generalizations of the injectivity were obtained, e.g., *principally injectivity*. In [2], V. Camillo introduced the definition of principally injective modules by calling a right R-module M is *principally injective* if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M.

In [10], W. K. Nicholson and M. F. Yousif studied to the structure of principally injective rings. They gave some applications of these rings and modules. A ring R is called *right principally injective* if every R-homomorphism from a principal right ideal of R to R can be extended to an R-homomorphism from R to R.

In [11], W. K. Nicholson, J. K. Park and M. F. Yousif introduced the definition of principally quasi injective modules by calling a right R-module M is *principally quasi-injective* if every R-homomorphism from a principal submodule of M to M can be extended to an R-endomorphism of M.

In [12], N. V. Sanh, K. P. Shum, S. Dhompomgsa and S. Wongwai introduced the definitions of quasi principally injective modules. A right R-module M is *quasi-principally injective* if every R-homomorphism from an M-Cyclic submodule of M to M can be extended to M.

In [19], Z. Zhanmin introduced the definitions of pseudo principally quasi injective modules. A right R-module M is *pseudo principally quasi-injective* if every R-monomorphism from a principal submodule of M to M can be extended to M.

1.2 Purpose of the Study

In this thesis, we have the purposes of study which are to extend concept of the previous works and to generalize new concepts which are :

1.2.1 To extend the concept of *principally injective modules*.

1.2.2 To generalize the concept of *principally quasi injective modules* and *quasi principally injective modules*.

1.2.3 To establish and extend some new concepts which is *pseudo principally quasi-injective modules* [19].

1.3 Research Questions and Hypothesis

We are interested in seeing to extend the characterizations and properties which remain valid from these previous concepts which can be extended from *principally injective modules* [2], *principally-injective rings* [10], *principally quasi-injective modules* [11], *quasi-principally injective modules* [12], and *pseudo principally quasi-injective modules* [19].

In this research, we introduce the definition of *pseudo principally quasi-injective modules* and *pseudo quasi-principally injective modules* and give characterizations and properties of these modules which are extended from the previous works. By let *M* be a right *R*-module. A right *R*-module *N* is called *pseudo principally M-injective* if every *R*-monomorphism from a principal submodule of *M* to *N* can be extended to an *R*-homomorphism from *M* to *N*. Dually, a right *R*-module *M* is called *pseudo principally quasi-injective* if it is *pseudo principally M-injective.* And a right *R*-module *N* is called *pseudo M-principally injective* if every *R*-monomorphism from an *M*-cyclic submodule of *M* to *N* can be extended to an *R*-homomorphism from *M* to *N*. Dually, a right *R*-module *M* is called *pseudo quasi-principally injective* if it is *pseudo M-principally injective*. Many of results in this research are extended from *principally-injective rings* [10], *principally quasi-injective modules* [11], *quasi-principally injective modules* [12], *endomorphism ring of semi-injective module* [16], and *pseudo principally quasi-injective modules* [19].

1.4 Theoretical Perspective

In this thesis, we use many of the fundamental theories which are concerned to the rings and modules research. By the concerned theories are :

1.4.1 The fundamental of algebra theories.

1.4.2 The basic properties of rings and modules theory.

1.5 Delimitations and Limitations of the Study

For this thesis, we have the scopes and the limitations of studying which are concerned to the previous works which are:

1.5.1 To extend the concept of *principally quasi injective modules* and *quasi principally injective modules*.

1.5.2 To extend the concept of *pseudo principally quasi-injective modules* and *pseudo quasi-principally injective modules*.

1.5.3 To characterize the concept in 1.5.2 and find some new properties.

1.6 Significance of the Study

The advantage of education and studying in this research, we can improve and develop the concepts and knowledge in the algebra and modules research field.

CHAPTER 2

LITERATURE REVIEW

In this chapter we give notations, definitions and fundamental theories of the modules and rings theory which are used in this thesis.

2.1 Rings, Modules, Submodules and Endomorphism Rings

This section is assembled summary of various notations, terminology and some background theories which are concerned and used for this thesis.

2.1.1 Definition. [15] By a *ring* we mean a nonempty set R with two binary operations $+$ and $\cdot$, called *addition* and *multiplication* (also called *product*), respectively, such that

(1) $(R, +)$ is an additive abelian group.

(2) $(R, \cdot)$ is a multiplicative semigroup.

(3) Multiplication is distributive (on both sides) over addition; that is, for all $a, b, c \in R$, $a\cdot(b+c) = a\cdot b+a\cdot c$ and $(a+b)\cdot c = a\cdot c+b\cdot c$.

The two distributive laws are respectively called the *left distributive* law and the *right distributive* law.

A *commutative ring* is a ring R in which multiplication is commutative; i.e. if $a\cdot b = b\cdot a$ for all $a, b \in R$. If a ring is not commutative it is called *noncommutative*.

A *ring with unity* is a ring R in which the multiplicative semigroup $(R, \cdot)$ has an identity element; that is, there exists $e \in R$ such that $ea = a = ae$ for all $a \in R$. The element e is called *unity* or the *identity* element of R. Generally, the unity or identity element is denoted by 1.

In this thesis, R will be an associative ring with identity.

2.1.2 Definition. [15] A nonempty subset I of a ring R is called an *ideal* of R if

(1) $a, b \in I$ implies $a - b \in I$.

(2) $a \in I$ and $r \in R$ imply $ar \in I$ and $ra \in I$.

2.1.3 Definition. [14] A subgroup I of $(R, +)$ is called a *left ideal* of R if $RI \subset I$, and a *right ideal* if $IR \subset I$.

2.1.4 Definition. [15] A right ideal I of a ring R is called *principal* if $I = aR$ for some $a \in R$.

2.1.5 Definition. [15] Let R be a ring, M an additive abelian group and $(m, r) \mapsto mr$, a mapping of $M \times R$ into M such that

(1) $mr \in M$

(2) $(m_1 + m_2)r = m_1r + m_2r$

(3) $m(r_1 + r_2) = mr_1 + mr_2$

(4) $(mr_1)r_2 = m(r_1r_2)$

(5) $m\cdot 1 = m$

for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. Then M is called a *right R-module*, often written as M_R.

Often mr is called the *scalar multiplication* or just *multiplication* of m by r on right. We define left R-module similarly.

2.1.6 Definition. [14] Let M be a right R-module. A subgroup N of $(M, +)$ is called a *submodule* of M if N is closed under multiplication with elements in R, that is $nr \in N$ for all $n \in N$, $r \in R$. Then N is also a right R-module by the operations induced from M :

$$N \times R \to N, (n, r) \mapsto nr, \text{ for all } n \in N, r \in R.$$

2.1.7 Proposition. *A subset N of an R-module M is a submodule of M if and only if*

(1) $0 \in N$.

(2) $n_1, n_2 \in N$ *implies* $n_1 - n_2 \in N$.

(3) $n \in N, r \in R$ *implies* $nr \in N$.

Proof. See [16, Lemma 5.3]. □

2.1.8 Definition. [1] Let M be a right R-module and let K be a submodule of M. Then the set of cosets

$$M/K = \left\{ x + K \,\middle|\, x \in M \right\}$$

is a right R-module relative to the addition and scalar multiplication defined via

$$(x+K) + (y+K) = (x+y) + K \quad \text{and} \quad (x+K)r = xr + K.$$

The additive identity and inverses are given by

$$K = 0 + K \quad \text{and} \quad -(x+K) = -x + K.$$

The module M/K is called (the *right R-factor module of*) *M modulo K* or the *factor module of M by K*.

2.1.9 Definition. [14] Let M and N be right R-modules. A function $f : M \to N$ is called an (R-*module*) *homomorphism* if for all $m, m_1, m_2 \in M$ and $r \in R$

$$f(m_1 r + m_2) = f(m_1)r + f(m_2).$$

Equivalently, $f(m_1 + m_2) = f(m_1) + f(m_2)$ and $f(mr) = f(m)r$.

The set of R-homomorphisms of M in N is denoted by $Hom_R(M,N)$. In particular, with this addition and the composition of mappings, $Hom_R(M,M) = End_R(M)$ becomes a ring, called the *endomorphism ring* of M and $f \in End_R(M)$ is called *an R-endomorphism.* [14, 6.4]

2.1.10 Definition. [1] Let $f : M \to N$ be an R-homomorphism. Then

(1) f is called R-*monomorphism* (or R-monic) if f is injective (one-to-one).

(2) f is called R-*epimorphism* (or R-epic) if f is surjective (onto).

(3) f is called R-*isomorphism* if f is bijective (one-to-one and onto).

Two modules M and N are said to be R-*isomorphic*, abbreviated $M \cong N$ in case there is an R-*isomorphism* $f : M \to N$.

2.1.11 Definition. [1] Let K be a submodule of M. Then the mapping $\eta_K : M \to M/K$ from M onto the factor module M/K defined by

$$\eta_K(x) = x + K \in M/K \qquad (x \in M)$$

is seen to be an R-epimorphism with kernel K. We call η_K the *natural epimorphism of M onto M/K.*

2.1.12 Definition. [1] Let $A \subset B$. Then the function $\iota = \iota_{A \subset B} : A \longrightarrow B$ defined by $\iota = (1_{B|A}) : a \mapsto a$ for all $a \in A$ is called the *inclusion map* of A in B. Note that if $A \subset B$ and $A \subset C$, and if $B \neq C$, then $\iota_{A \subset B} \neq \iota_{A \subset C}$. Of course $1_A = \iota_{A \subset A}$.

2.1.13 Definition. [15] Let M and N be right R-modules and let $f : M \longrightarrow N$ be an R-homomorphism. Then the set

$Ker(f) = \{ x \in M \mid f(x) = 0 \}$ is called the *kernel* of f

and

$f(M) = \{ f(x) \in N \mid x \in M \}$ is called the *homomorphic image* (or simply *image*) of M under f and is denoted by $Im(f)$.

2.1.14 Proposition. *Let M and N be right R-modules and let $f : M \longrightarrow N$ be an R-homomorphism. Then*

(1) *$Ker(f)$ is a submodule of M.*

(2) *$Im(f) = f(M)$ is a submodule of N.*

Proof. See [14, 6.5]. □

2.1.15 Proposition. *Let M and N be right R-modules and let $f : M \longrightarrow N$ be an R-isomorphism. Then the inverse mapping $f^{-1}: N \longrightarrow M$ is an R-isomorphism.*

Proof. See [15, Chapter 14, 3]. □

2.1.16 Definition. [20] A submodule K of the module M is fully invariant in M if $f(K) \subset K$ for every endomorphism f of M.

2.2 Essential and Superfluous Submodules

In this section, we give the definitions of essential and superfluous submodules and some theories which are used in this thesis.

2.2.1 Definition. [14] A submodule K of M is called *essential* (or *large*) in M, abbreviated $K \subset^e M$, if for every submodule L of M, $K \cap L = 0$ implies $L = 0$.

2.2.2 Definition. [14] A submodule K of M is called *superfluous* (or *small*) in M, abbreviated $K \ll M$, if for every submodule L of M, $K + L = M$ implies $L = M$.

2.2.3 Proposition. *Let M be a right R-module with submodules $K \subset N \subset M$ and $H \subset M$. Then*

(1) $K \subset^e M$ *if and only if* $K \subset^e N$ *and* $N \subset^e M$.

(2) $K \cap H \subset^e M$ *if and only if* $K \subset^e M$ *and* $H \subset^e M$.

Proof. See [1, Proposition 5.16]. □

2.2.4 Proposition. *A submodule $K \subset M$ is essential in M if and only if for each* $0 \neq x \in M$ *there exists an* $r \in R$ such that $0 \neq xr \in K$.

Proof. See [1, Proposition 5.19]. □

2.2.5 Definition. [7] Let R be a ring and M is a right R-module. M is *co-Hopfian* if any injective endomorphism of M is an isomorphism. A right R-module M is weakly *co-Hopfian* if any injective endomorphism f of M is essential; that is, $f(M) \subset^e M$.

2.2.6 Definition. [1] A nonzero module M is *uniform* if every non-zero submodule of M is essential in M.

2.3 Annihilators and Singular Modules

In this section, we give the definitions of annihilators, singular modules and some theories which are used in this thesis.

2.3.1 Definition. [1] Let M be a right (resp. left) R-module. For each $X \subset M$, the *right* (resp. *left*) *annihilator* of X in R is defined by

$$r_R(X) = \{ r \in R \mid xr = 0, \forall x \in X \} \text{ (resp. } l_R(X) = \{ r \in R \mid rx = 0, \forall x \in X \}).$$

For a singleton $\{x\}$, we usually abbreviated to $r_R(x)$ (resp. $l_R(x)$).

2.3.2 Proposition. *Let M be a right R-module, let X and Y be subsets of M and let A and B be subsets of R. Then*

(1) $r_R(X)$ *is a right ideal of* R.

(2) $X \subset Y$ *imples* $r_R(Y) \subset r_R(X)$.

(3) $A \subset B$ *imples* $l_M(B) \subset l_M(A)$.

(4) $X \subset l_M r_R(X)$ *and* $A \subset r_R l_M(A)$.

Proof. See [1, Proposition 2.14 and Proposition 2.15]. □

2.3.3 Proposition. *Let M and N be right R-modules and let $f : M \to N$ be a homomorphism. If N' is an essential submodule of N, then $f^{-1}(N')$ is an essential submodule of M.*

Proof. See [4, Lemma 5.8(a)]. □

2.3.4 Proposition. *Let M be a right R-module over an arbitrary ring R, the set*

$$Z(M) = \{ x \in M \mid r_R(x) \text{ is essential in } R_R \}$$

is a submodule of M.

Proof. See [4, Lemma 5.9]. □

2.3.5 Definition. [4] The submodule $Z(M) = \{ x \in M \mid r_R(x) \text{ is essential in } R_R \}$ is called the *singular submodule* of M. The module M is called a *singular module* if $Z(M) = M$. The module M is called *a nonsingular module* if $Z(M) = 0$.

2.4 Maximal and Minimal Submodules

In this section, we give the definitions and some properties of maximal submodules, minimal (or simple) submodules and some theories which are used in this thesis.

2.4.1 Definition. [14] A right R-module M is called *simple* if $M \neq 0$ and M has no submodules except 0 and M.

2.4.2 Definition. [14] A submodule K of M is called *maximal submodule* of M if $K \neq M$ and it is not properly contained in any proper submodules of M, i.e. K is *maximal in M* if, $K \neq M$ and for every $A \subset M$, $K \subset A$ implies $K = A$.

2.4.3 Definition. [14] A submodule N of M is called *minimal* (or *simple*) *submodule* of M if $N \neq 0$ and it has no non zero proper submodules of M, i.e. N is *minimal* (or *simple*) *in M* if $N \neq 0$ and for every nonzero submodules A of M, $A \subset N$ implies $A = N$.

2.4.4 Proposition. *Let M and N be right R-modules. If $f : M \to N$ is an epimorphism with $Ker(f) = K$, then there is a unique isomorphism $\sigma : M/K \to N$ such that $\sigma(m+K) = f(m)$ for all $m \in M$.*

Proof. See [1, Corollary 3.7]. □

2.4.5 Proposition. *Let K be a submodule of M. A factor module M/K is simple if and only if K is a maximal submodule of M.*

Proof. See [1, Corollary 2.10]. □

2.5 Injective and Projective Modules

In this section, we give the definitions of the injective modules and some theories which are used in this thesis.

2.5.1 Definition. [1] Let M be a right R-module. A right R-module U is called *injective relative to M* (or *U is M-injective*) if for every submodule K of M, for every homomorphism $\varphi : K \to U$ can be extended to a homomorphism $\alpha : M \to U$.

A right R-module U is said to be *injective* if it is M-injective for every right R-module M.

2.5.2 Proposition. *The following statements about a right R-module U are equivalent* :

(1) *U is injective*;

(2) *U is injective relative to R*;

(3) *For every right ideal $I \subset R_R$ and every homomorphism $h : I \longrightarrow U$ there exists an $x \in U$ such that h is left multiplicative by x*

$$h(a) = xa \text{ for all } a \in I.$$

Proof. See [1, 18.3, Baer's Criterion]. □

2.5.3 Definition. [1] Let M be a right R-module. A right R-module U is called *projective relative to M* (or *U is M-projective*) if for every N_R, every epimorphism $g : M_R \longrightarrow N_R$, for every homomorphism $\gamma : U_R \longrightarrow N_R$ can be lifted to an R-homomorphism $\hat{\gamma} : U \longrightarrow M$.

A right R-module U is said to be *projective* if it is projective for every right R-module M.

2.5.4 Proposition. *Every right* (*resp. left*) *R-module can be embedded in an injective right* (*resp. left*) *R-module*.

Proof. See [1, Proposition 18.6]. □

2.6 Direct Summands and Product of Modules

Given two modules M_1 and M_2 we can construct their Cartesian product $M_1 \times M_2$. The structure of this product module is then determined "co-ordinatewise" from the factors $M_1 \times M_2$. For this section we give the definitions of direct summand, the projection and the injection maps, product of modules and some theories which are used in this thesis.

2.6.1 Definition. [1] Let M be a right R-module. A submodule X of M is called a *direct summand* of M if there is a submodule Y of M such that $X \cap Y = 0$ and $X + Y = M$. We write $M = X \oplus Y$; such that Y is also a *direct summand*.

2.6.2 Definition. [1] Let M_1 and M_2 be R-modules. Then with their products module $M_1 \times M_2$ are associated the natural injections and projections

$$\varphi_j : M_j \longrightarrow M_1 \times M_2 \qquad \text{and} \qquad \pi_j : M_1 \times M_2 \longrightarrow M_j$$

$(j = 1, 2)$, are defined by

$$\varphi_1(x_1) = (x_1, 0), \qquad \varphi_2(x_2) = (0, x_2)$$

and

$$\pi_1(x_1, x_2) = x_1, \qquad \pi_2(x_1, x_2) = x_2.$$

Moreover, we have

$$\pi_1 \varphi_1 = 1_{M_1} \qquad \text{and} \qquad \pi_2 \varphi_2 = 1_{M_2}.$$

2.6.3 Definition. [1] Let A be a direct summand of M with complementary direct summand B, so $M = A \oplus B$. Then

$$\pi_A : a + b \mapsto a \qquad (a \in A, b \in B)$$

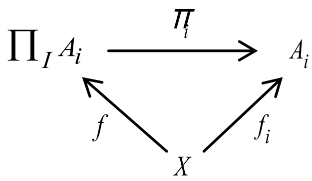
defines an epimorphism $\pi_A : M \longrightarrow A$ is called *the projection of M on A along B*.

2.6.4 Definition. [14] Let $\{A_i, i \in I\}$ be a family of objects in the category C. An object P in C with morphisms $\{\pi_i : P \longrightarrow A_i\}$ is called the *product* of the family $\{A_i, i \in I\}$ if :

For every family of morphisms $\{f_i : X \longrightarrow A_i\}$ in the category C, there is a unique morphism $f : X \longrightarrow P$ with $\pi_i f = f_i$ for all $i \in I$.

For the object P, we usually write $\prod_{i \in I} A_i$, $\prod_I A_i$ or $\prod A_i$. If all A_i are equal to A, then we put $\prod_I A_i = A^I$.

The morphism π_i are called the *i-projections* of the product. The definition can be described by the following commutative diagram :



2.6.5 Definition. [14] Let $\{ M_i , i \in I \}$ be a family of *R*-modules and ($\prod_{i \in I} M_i$, π_i) the product of the M_i . For *m*, $n \in \prod_{i \in I} M_i$, $r \in R$, using

$$\pi_i (m + n) = \pi_i (m) + \pi_i (n) \quad \text{and} \quad \pi_i(mr) = \pi_i (m)r,$$

a right *R*-module structure is defined on $\prod_{i \in I} M_i$ such that the π_i are homomorphisms. With this structure ($\prod_{i \in I} M_i$, π_i) is the product of the $\{ M_i , i \in I \}$ in *R*-module.

2.6.6 Proposition. *Properties*:

(1) *If* $\{ f_i : N \rightarrow M_i , i \in I \}$ *is a family of morphisms, then we get the map*

$$f : N \rightarrow \prod_{i \in I} M_i \quad \text{such that} \quad n \mapsto \big(f_i(n) \big)_{i \in I}$$

and $Ker(f) = \bigcap_I Ker(f_i)$ *since* $f(n) = 0$ *if and only if* $f_i(n) = 0$ *for all* $i \in I$.

(2) *For every* $j \in I$, *we have a canonical embedding*

$$\varepsilon_j : M_j \rightarrow \prod_{i \in I} M_i , \quad \text{such that} \quad m_j \mapsto (m_j \delta_{ji})_{i \in I} , m_j \in M_j ,$$

with $\varepsilon_j \pi_j = 1_{M_j}$, *i.e.* π_j *is a retraction and* ε_j *a coretraction.*

This construction can be extended to larger subsets of I : *For a subset* $A \subset I$ *we form the product* $\prod_{i \in A} M_i$ *and a family of homomorphisms*

$$f_j : \prod_{i \in A} M_i \rightarrow M_j , \qquad f_j = \begin{cases} \pi_j & for\ j \in A, \\ 0 & for\ j \in I - A. \end{cases}$$

Then there is a unique homomorphism

$$\varepsilon_A : \prod_{i \in A} M_i \rightarrow \prod_{i \in I} M_i \text{ with } \varepsilon_A \pi_j = \begin{cases} \pi_j & \text{for } j \in A, \\ 0 & \text{for } j \in I - A. \end{cases}$$

The universal property of $\prod_{i \in A} M_i$ *yields a homomorphism*

$$\pi_A : \prod_{i \in I} M_i \rightarrow \prod_{i \in A} M_i \text{ with } \pi_A \pi_j = \pi_j \text{ for } j \in I.$$

Together this implies $\varepsilon_A \pi_A \pi_j = \varepsilon_A \pi_j = \pi_j$ *for all* $j \in I$, *and by the properties of the product* $\prod_{i \in A} M_i$,

we get $\varepsilon_A \pi_A = 1_{M_A}$.

Proof. See [14, 9.3, Properties (1), (2)] □

2.7 Split Homomorphisms

Throughout this thesis, all rings R are associative with identity and all modules are unitary right R-modules. A submodule X of M is called a direct summand of M if there is a submodule Y of M such that $X \cap Y = 0$ and $X + Y = M$. We will write $M = X \oplus Y$.

2.7.1 Lemma. Let $f : M \rightarrow N$ and $g : N \rightarrow M$ be homomorphisms such that $fg = 1_N$. Then f is an epimorphism, g is a monomorphism and $M = \text{Ker}(f) \oplus \text{Im}(g)$.

Proof. See [1, Lemma 5.1]. □

If $f : M \rightarrow N$ and $g : N \rightarrow M$ be homomorphisms such that $fg = 1_N$, then we say that f is a *split epimorphism* (or *splits*), and we write

$$M \xrightarrow{\ f\ } \!\!\!\!\!\oplus\; N \longrightarrow 0 ;$$

and we say that g is a *split monomorphism* (or *splits*), and we write

$$0 \longrightarrow N \xrightarrow{\ g\ } \!\!\!\!\!\oplus\; M ;$$

A short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{\ f\ } M \xrightarrow{\ g\ } M_2 \longrightarrow 0$$

is *split* (or *splits*) if f is a split monomorphism and g is a split epimorplism.

2.7.2 Proposition *The following statements about a short exact sequence* $0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$ *in M are equivalent* :

(1) *The sequence is splits*;

(2) *The monomorphism* $f : M_1 \rightarrow M$ *is split*;

(3) *The epimorphism* $g : M \rightarrow M_2$ *is split*;

(4) *Im f = Ker g is a direct summand of M*;

(5) *Every homomorphism* h: $M_1 \rightarrow N$ *factors through f*;

(6) *Every homomorphism* $h : N \rightarrow M_2$ *factors though g*.

Proof. See [1, Propostion 5.2]. □

2.8 Generated and Cogenerated Classes

In this section, we give some definitions and theories of the generated and cogenerated classes which are concerned in this thesis.

2.8.1 Definition. [14] A subset X of a right R-module M is called a *generating set* of M if $XR = M$. We also say that *X generates M* or M is *generated by X*. If there is a finite generating set in M, then M is called *finitely generated*.

2.8.2 Definition. [1] Let $\mathcal{U}$ be a class of right R-modules. A module M is (*finitely*) *generated by* $\mathcal{U}$ (or $\mathcal{U}$ (*finitely*) *generates M*) if there exists an epimorphism

$$\bigoplus_{i \in I} U_i \rightarrow M$$

for some (finite) set I and $U_i \in \mathcal{U}$ for every $i \in I$.

If $\mathcal{U} = \{U\}$ is a singleton, then we say that M is (*finitely*) *generated by* $\mathcal{U}$ or (*finitely*) *U-generates*; this means that there exists an epimorphism

$$U^{(I)} \rightarrow M$$

for some (finite) set I.

2.8.3 Proposition. *If a module M has a generating set $L \subset M$, then there exists an epimorphism*

$$R^{(L)} \to M$$

Moreover, M is finitely R-generated if and only if M is finitely generated.

Proof. See [1, Theorem 8.1]. □

2.8.4 Definition. [18] Let M be a right R-module. A submodule N of M is said to be an *M-cyclic submodule* of M if it is the image of an endomorphism of M.

2.8.5 Definition. [1] Let $\mathcal{U}$ be a class of right R-modules. A module M is (*finitely*) *cogenerated by* $\mathcal{U}$ (or $\mathcal{U}$ (*finitely*) *cogenerates* M) if there exists a monomorphism

$$M \to \prod_{i \in I} U_i$$

for some (finite) set I and $U_i \in \mathcal{U}$ for every $i \in I$.

If $\mathcal{U} = \{U\}$ is a singleton, then we say that a module M is (*finitely*) *cogenerated by* $\mathcal{U}$ or (*finitely*) *U-cogenerates*; this means that there exists a monomorphism

$$M \to U^I$$

for some (finite) set I.

2.9 The Trace and Reject

In this section, we give some definitions and theories of the trace and reject which are concerned in this thesis.

2.9.1 Definition. [1] Let $\mathcal{U}$ be a class of right R-modules. The *trace* of $\mathcal{U}$ in M and the *reject* of $\mathcal{U}$ in M are defined by

$$Tr_M(\mathcal{U}) = \sum \{ Im(h) \mid h : U \to M \text{ for some } U \in \mathcal{U} \}$$

and

$$Rej_M(\mathcal{U}) = \bigcap \{ Ker(h) \mid h : M \to U \text{ for some } U \in \mathcal{U} \}.$$

If $\mathcal{U} = \{ U \}$ is a singleton, then the trace of $\mathcal{U}$ in *M* and the reject of $\mathcal{U}$ in *M* are in the form

$$Tr_M(U) = \sum \{ Im(h) \mid h \in Hom_R(U, M) \}$$

and

$$Rej_M(U) = \bigcap \{ Ker(h) \mid h \in Hom_R(M, U) \}.$$

2.9.2 Proposition. *Let* $\mathcal{U}$ *be a class of right R-modules and let M be a right R-module. Then*

(1) $Tr_M(\mathcal{U})$ *is the unique largest submodule L of M generated by* $\mathcal{U}$;

(2) $Rej_M(\mathcal{U})$ *is the unique smallest submodule K of M such that* M/K *is cogenerated by* $\mathcal{U}$.

Proof. See [1, Proposition 8.12]. □

2.10 Socle and Radical of Modules

In this section, we give some definitions and theories of the socle and radical of modules which are used in this thesis.

2.10.1 Definition. [14] Let *M* be a right *R*-module. The *socle* of *M*, $Soc(M)$, we denote the sum of all simple submodules of *M*. If there are no simple submodules in *M* we put $Soc(M) = 0$.

2.10.2 Definition. [14] Let *M* be a right *R*-module. The *radical* of *M*, $Rad(M)$, we denote the intersection of all maximal submodules of *M*. If *M* has no maximal submodules we set $Rad(M) = M$.

2.10.3 Proposition. *Let* $\mathcal{E}$ *be the class of simple R-modules and let M be an R-module. Then*

$$\begin{aligned} Soc(M) &= Tr_M(\mathcal{E}) \\ &= \bigcap \{ L \subset M \mid L \text{ is essential in } M \}. \end{aligned}$$

Proof. See [14, 21.1]. □

2.10.4 Proposition. *Let $\mathcal{E}$ be the class of simple R-modules and let M be an R-module. Then*

$$Rad(M) = Rej_M(\mathcal{E}) = \sum\{ L \subset M \mid L \text{ is superfluous in } M \}.$$

Proof. See [14, 21.5]. □

2.10.5 Proposition. *Let M be a right R-module. A right R-module M is finitely generated if and only if $Rad(M) \ll M$ and $M/Rad(M)$ is finitely generated.*

Proof. See [14, 21.6, (4)]. □

2.10.6 Proposition. *Let M be a right R-module. Then $Soc(M) \subset^e M$ if and only if every non-zero submodule of M contains a minimal submodule.*

Proof. See [1, Corollary 9.10]. □

2.10.7 Corollary. [7] Let M_R is weakly co-Hopfian and *f* is an injective endomorphism of *M*, then:

(1) $N \subset^e M$ if and only if $f(N) \subset^e M$ *and* $f^{-1}(N) \subset^e M$.

(2) Soc(*N*) = $\bigcap f(N) = \bigcap f^{-1}(N)$, where *N* runs through the set of all essential submodules of *M*.

2.11 The Radical of a Ring and Local Rings

In this section, we give some definitions and theories of the radical of a ring and local rings which are used in this thesis.

2.11.1 Definition. [1] Let *R* be a ring. The radical $Rad(R_R)$ of R_R is an (two side) ideal of *R*. This ideal of *R* is called the (*Jacobson*) *radical* of *R*, and we usually abbreviated by

$$J(R) = Rad(R_R).$$

2.11.2 Definition. [1] Let *R* be a ring. An element $x \in R$ is called *right* (*left*) *quasi-regular* if $1 - x$ has a right (resp. left) inverse in *R*.

An element $x \in R$ is called *quasi-regular* if it is right and left quasi-regular.

A subset of R is said to be (*right*, *left*) *quasi-regular* if every element in it has the corresponding property.

2.11.3 Proposition. *Given a ring R for each of the following subsets of R is equal to the radical $J(R)$ of R.*

(J_1) *The intersection of all maximal right (left) ideals of R;*

(J_2) *The intersection of all right (left) primitive ideals of R;*

(J_3) $\{ x \in R \mid rxs$ *is quasi-regular for all* $r, s \in R \}$;

(J_4) $\{ x \in R \mid rx$ *is quasi-regular for all* $r \in R \}$;

(J_5) $\{ x \in R \mid xs$ *is quasi-regular for all* $s \in R \}$;

(J_6) *The union of all the quasi-regular right (left) ideals of R;*

(J_7) *The union of all the quasi-regular ideals of R;*

(J_8) *The unique largest superfluous right (left) ideals of R;*

Moreover, (J_3), (J_4), (J_5), (J_6) and (J_7) also describe the radical $J(R)$ if "quasi-regular" is replaced by "right quasi-regular" or by "left quasi-regular".

Proof. See [1, Theorem 15.3]. □

2.11.4 Proposition. *Let R be a ring with radical $J(R)$. Then for every right R-module M,*

$$J(R)M_R \subset Rad(M_R).$$

If R is semisimple modulo its radical, then for every right R-module,

$$J(R)M_R = Rad(M_R)$$

and $M/J(R)M_R$ is semisimple.

Proof. See [1, Corollary 15.18]. □

2.11.5 Definition. A ring R is said to be *local* if the set of non-inverrtible elements of R is closed under addition.

2.11.6. Proposition. *For a ring R the following statements are equivalent*:

(1) *R is a local ring*;

(2) *R has a unique maximal right ideal*;

(3) *J*(*R*) *is a maximal right ideal*;

(4) *The set of elements of R without right inverses is closed under addition*;

(5) $J(R) = \{ x \in R \mid Rx \neq R \}$;

(6) $R/J(R)$ *is a division ring*;

(7) $J(R) = \{ x \in R \mid x$ *is not invertible* $\}$;

(8) *If* $x \in R$, *then either* x *or* $1-x$ *is invertible*.

Proof. See [1, Proposition 15.15] □

2.12 Von Neumann Regular Rings

In this section, we give some definitions and theories of Von Neumann regular rings which are used in this thesis.

2.12.1 Definition. A ring *R* is von Neumann regular if $a \in aRa$ for each $a \in R$.

2.12.2 Proposition. *The following statements are equivalent for a ring R*:

(1) *R is von Neumann regular*;

(2) *Every principal right ideal is a direct summand*;

(3) *Every finitely generated right is a direct summand*.

Proof. See [3, 3.10].

2.12.3 Proposition. Let *M* be a right *R*-module and $S = End(M_R)$. Then the following statements are equivalent:

(1) *S is von Neumann regular*;

(2) *Im*(f) *and Ker*(f) *are direct summand of M for every* $f \in S$.

Proof. See [14, 37.7]. □

CHAPTER 3

RESEARCH RESULT

In this chapter, we present the results of pseudo principally quasi-injective modules and pseudo quasi-principally injective modules.

3.1 Pseudo Principally Quasi-injective Modules

3.1.1 Definition. [19] Let M be a right R-module. A right R-module N is called *pseudo principally M-injective* (briefly, PP-M-injective) if, every R-monomorphism from a principal submodule of M to N can be extended to M. The module M is called *pseudo principally quasi-injective* (briefly, PPQ-injective) if, it is pseudo principally M-injective.

3.1.2 Proposition. *Let M and N_i ($i = 1,2,\ldots,n$) be right R-modules. If $\bigoplus_{i=1}^{n} N_i$ is PP-M-injective, then N_i is PP-M-injective for each $i = 1,2,\ldots,n$.*

Proof. Let $i \in \{1, 2, ..., n\}$. To show that N_i is PP-M-injective. Let $m \in M$ and $\varphi : mR \longrightarrow N_i$ be an R-monomorphism. Let $\pi_i : \bigoplus_{i=1}^{n} N_i \longrightarrow N_i$ be the i-th projection map and $\varphi_i : N_i \longrightarrow \bigoplus_{i=1}^{n} N_i$ be the i-th injection map. Since $\varphi_i\varphi$ is an R-monomorphism, there exists an R-homomorphism $\hat{\varphi} : M \longrightarrow \bigoplus_{i=1}^{n} N_i$ such that $\hat{\varphi}\iota = \varphi_i\varphi$ where $\iota : mR \longrightarrow M$ is the inclusion map. Then $\pi_i\hat{\varphi}\iota = \pi_i\varphi_i\varphi$. Then by Definition 2.6.2, $\pi_i\hat{\varphi}\iota = \varphi$. Hence $\pi_i\hat{\varphi}$ is an extension of φ. □

3.1.3 Lemma. *Let B be a principal submodule of M. If B is PP-M-injective, then it is a direct summand of M.*

Proof. Let $B = mR$, $m \in M$ and B is PP-M-injective. Let $\iota : mR \longrightarrow M$ be the inclusion map and $1_{mR} : mR \longrightarrow mR$ be the identity map. Since mR is PP-M-injective, there exists an R-homomorphism $\hat{\varphi} : M \longrightarrow mR$ such that $\hat{\varphi}\iota = 1_{mR}$. Then we see that the short exact sequence $0 \rightarrow mR \xrightarrow{\iota} M$ splits.

Then by Proposition 2.7.2, mR = Im(ι) is a direct summand of M. This shows that B is a direct summand of M. □

3.1.4 Lemma. *Let M be PPQ-injective. If A is a direct summand of M, then A is PP-M-injective.*

Proof. Let A be a direct summand of M. Let $m \in M$ and $\alpha : mR \longrightarrow A$ be an R-monomorphism. Let $\varphi : A \longrightarrow M$ be the injection map. Then $\varphi\alpha : mR \longrightarrow M$ is an R-monomorphism. Since M is PPQ-injective, there exists an R-homomorphism $\hat{\alpha} : M \longrightarrow M$ such that $\varphi\alpha = \hat{\alpha}\iota$ where $\iota : mR \longrightarrow M$ is the inclusion map. Let $\pi : M \longrightarrow A$ be the projection map. Then $\pi\varphi\alpha = \pi\hat{\alpha}\iota$. Since by Proposition 2.6.6, $\pi\varphi = 1_A$, $\alpha = \pi\hat{\alpha}i$. Therefore $\pi\hat{\alpha}$ is an extension of α. This shows that A *is PP-M*-injective. □

A right R-module M is called *co-Hopfian* [7] if any injective endomorphism of M is an isomorphism. A right R-module M is called *weakly co-Hopfian* if any injective endomorphism f of M is essential; that is, $f(M) \subset^e M$. A submodule N of M is called a fully invariant submodule of M if $s(N) \subset N$ for every $s \in S = End_R(M)$.

3.1.5 Proposition. *Let M be a principal and PPQ-injective module.*

(1) *If M is weakly co-Hopfian, then M is co-Hopfian.*

(2) *For a fully invariant essential submodule X of M, if X is weakly co-Hopfian, then M is weakly co-Hopfian.*

(3) *If X is a principal and essential submodule of M and M is weakly co-Hopfian, then X is weakly co-Hopfian.*

Proof. (1) Let M be a weakly co-Hopfian module. Let $f : M \longrightarrow M$ be an R-monomorphim. Since f is monic, $f(M) \cong M$. We must show that $f(M)$ is PP-M-injective. Let $m \in M$ and $\alpha : mR \longrightarrow f(M)$ be an R-monomorphism. Let $\sigma : f(M) \longrightarrow M$ be the R-isomorphism. Since M is PPQ-injective, there exists an R-homomorphism $\hat{\alpha} : M \longrightarrow M$ such that $\sigma\alpha = \hat{\alpha}\iota$ where $\iota : M \longrightarrow M$ is the inclusion map. Then $\sigma^{-1}\sigma\alpha = \sigma^{-1}\hat{\alpha}\iota$, so $\alpha = \sigma^{-1}\hat{\alpha}\iota$. Hence $f(M)$ is PP-M-injective. Since $f(M)$ is a principal submodule of M, by Lemma 3.1.3, $M = f(M) \oplus X$ for some submodule X of M. Thus $f(M) \cap X = 0$

and $M = f(M) + X$. Since M is weakly co-Hopfian, $f(M) \subset^e M$. Hence $X = 0$. Therefore $M = f(M)$. This shows that M is co-Hopfian.

(2) Let X be a fully invariant essential submodule of M and let X be weakly co-Hopfian. To show that M is weakly co-Hopfian. Let $f : M \to M$ be an R-monomorphism. Since X is fully invariant submodule of M, $f|_X : X \to X$ is an endomorphism of X. It follows that $f|_X : X \to X$ is an R-monomorphism. We must show that $f(M) \subset^e M$. Since X is weakly co-Hopfian, $f(X) \subset^e X$. Since $X \subset^e M$, $f(X) \subset^e M$. Since $f(X) \subset f(M) \subset M$ and $f(X) \subset^e M$, by Proposition 2.2.3 we have $f(M) \subset^e M$. Therefore M is weakly co-Hopfian.

(3) Let $X = mR$, for some $m \in M$, X be an essential submodule of M and let M be weakly co-Hopfian. To show that X is weakly co-Hopfian. Let $f : X \to X$ be an injective endomorphism of X. Since M is a PPQ-injective, there exists an R-homomorphism $g : M \to M$ such that $\iota f = g\iota$ where $\iota : X \to M$ is the inclusion map. Since X is an essential submodule of M and $Ker(g) \cap X = 0$, so by Definition 2.2.1, $Ker(g) = 0$. Hence g is an R-monomorphism, so $g(X) \subset^e M$ by Corollary 2.10.7. Since $f(X) = \iota f(X) = g\iota(X) = g(X)$, $f(X) \subset^e M$. Since $f(X) \subset X \subset M$, by Proposition 2.2.3 we have $f(X) \subset^e X$. Therefore X is weakly co-Hopfian. □

3.2 Pseudo Quasi-Principally injective Modules

3.2.1 Definition. Let M be a right R-module. A right R-module N is called *pseudo M-principally injective* (briefly, *PM-P*-injective) if, every R-monomorphism from an M-cyclic submodule of M to N can be extended to an endomorphism of M. The Module M is called *pseudo quasi-principally injective* (briefly, *PQ-P*-injective) if it is *PM-P*-injective.

3.2.2 Theorem. *Let M be a right R-module. Then M is PQ-P-injective if and only if $Ker(s) = Ker(t)$, $s, t \in S = End_R(M)$ implies $Ss = St$.*

Proof. ($\Rightarrow$) Let $s, t \in S$ with $Ker(s) = Ker(t)$. Define $\varphi : s(M) \to M$ by $\varphi(s(m)) = t(m)$ for every $m \in M$. We must show that φ is the well-defined. Let $s(m_1), s(m_2) \in s(M)$ such that $s(m_1) = s(m_2)$. Thus $s(m_1) - s(m_2) = 0$, so $s(m_1 - m_2) = 0$. Then $m_1 - m_2 \in Ker(s) = Ker(t)$, so $t(m_1 - m_2) = 0$. Hence $t(m_1) = t(m_2)$, so $\varphi(s(m_1)) = t(m_1) = t(m_2) = \varphi(s(m_2))$. Let $s(m_1), s(m_2) \in s(M)$ and $r \in R$.

Then $\varphi(s(m_1)r + s(m_2)) = \varphi(s(m_1r) + s(m_2)) = \varphi(s(m_1r + m_2)) = t((m_1r + m_2)) = t(m_1r) + t(m_2) = t(m_1)r + t(m_2) = \varphi(s(m_1))r + \varphi(s(m_2))$. This shows that φ is an R-homomorphism. Let $s(m_1)$, $s(m_2) \in s(M)$ such that $\varphi(s(m_1)) = \varphi(s(m_2))$. Then $t(m_1) = t(m_2)$, so $t(m_1 - m_2) = 0$. Thus $m_1 - m_2 \in Ker(t) = Ker(s)$, so $s(m_1 - m_2) = 0$. Hence $s(m_1) - s(m_2) = 0$, so $s(m_1) = s(m_2)$. This shows that φ is an R-monomorphism. Since M is pseudo quasi-principally injective and $s(M)$ is an M-cyclic submodule of M, there exists an R-homomorphism $\hat{\varphi}: M \to M$ such that $\varphi = \hat{\varphi}\iota$ where $\iota : s(M) \to M$ is the inclusion map. Thus $t = \varphi s = \hat{\varphi}\iota s = \hat{\varphi}s \in Ss$. Then $St \subset Ss$. Similarly, $Ss \subset St$, therefore $Ss = St$.

($\Leftarrow$) Let $s \in S$ and $\alpha : s(M) \to M$ be an R-monomorphism. Then $Ker(\alpha) = Ker(\iota)$. where $\iota : s(M) \to M$ is the inclusion map. Then by assumption, $S\alpha = S\iota$. We have $\alpha \in S\alpha$, so $\alpha \in S\iota$, write $\alpha = \beta\iota$, for some $\beta \in S$. This shows that M is pseudo quasi-principally injective. □

3.2.3 Theorem. *Let M be a PQ-P-injective module and* $s, t \in S = End_R(M)$.

(1) *If* $s(M)$ *embeds in* $t(M)$, *then* Ss *is an image of* St.

(2) *If* $s(M) \cong t(M)$, *then* $Ss \cong St$.

Proof. (1) Let $f : s(M) \to t(M)$ be an R-monomorphism. Let $\iota_1 : s(M) \to M$ and $\iota_2 : t(M) \to M$ be the inclusion maps. Since $\iota_2 f$ is an R-monomorphism and M is PQ-P-injective, there exists an R-homomorphism $\hat{f} : M \to M$ such that $\hat{f}\iota_1 = \iota_2 f$. Define $\sigma : St \to Ss$ by $\sigma(ut) = u\hat{f}s$ for every $u \in S$. To show that σ is well-defined. Let $ut = 0$. To show that $u\hat{f}s = 0$. Let $m \in M$. Since $\hat{f}\iota_1 = \iota_2 f$, $\hat{f}s(m) = \hat{f}\iota_1(s(m)) = \iota_2 f(s(m)) = fs(m)$ so $u\hat{f}s(m) = ufs(m)$. Since $fs(M) \subset t(M)$, $ufs(M) \subset ut(M) = 0$. Hence $ufs(M) = 0$ and so $u\hat{f}s(m) = 0$. To show that σ is a left S-homomorphism. Let ut, $vt \in St$ and let $g \in S$. Then $\sigma(gut + vt) = \sigma[(gu + v)t] = (gu + v)\hat{f}s = gu\hat{f}s + v\hat{f}s = g\sigma(ut) + \sigma(vt)$. Now we show that $Ker(\hat{f}s) = Ker(s)$. Let $x \in Ker(\hat{f}s)$. Then $\hat{f}s(x) = 0$. Then $fs(x) = 0$ so $s(x) = 0$ because f is monic. This shows that $Ker(\hat{f}s) \subset Ker(s)$. It is clear that $Ker(s) \subset Ker(\hat{f}s)$. Then $Ker(\hat{f}s) = Ker(s)$. Hence by Theorem 3.2.2 $Ss = S\hat{f}s$ so $s = u\hat{f}s$ for some $u \in S$, hence $s = u\hat{f}s = \sigma(ut) \in \sigma(St)$. It follows that $Ss = \sigma(St)$. This shows that σ is an S-epimorphism.

(2) Let $f : s(M) \to t(M)$ be an R-isomorphism. Let $\iota_1 : s(M) \to M$ and $\iota_2 : t(M) \to M$ be the inclusion maps. Since $\iota_2 f$ is an R-monomorphism and M is PQ-P-injective, there exists an R-homomorphism $\hat{f} : M \to M$ such that $\hat{f}\iota_1 = \iota_2 f$. Define $\sigma : St \to Ss$ by $\sigma(ut) = u\hat{f}s$ for every $u \in S$. The same argument as in (1), we show that σ is a left S-epimorphism. To show that σ is a left S-monomorphism. That is, show that $Ker(\sigma) = \{0\}$. ($\supset$) is clear. ($\subset$) Let $ut \in Ker(\sigma)$. Thus $\sigma(ut) = 0$, so $u\hat{f}s = 0$. Since $\hat{f}s(M) = t(M)$, $u\hat{f}s(M) = ut(M)$ hence $ut(M) = 0$. This shows that $ut = 0$. It follows that $Ker(\sigma) \subset \{0\}$. □

Clearly, every X-cyclic submodule of X is an M-cyclic submodule of M for every M-cyclic submodule X of M. Thus we have the following

3.2.4 Proposition. *N is PM-P-injective if and only if N is PX-P-injective for every M-cyclic submodule X of M.*

Proof. ($\Rightarrow$) Let $X = s(M)$ be an M-cyclic submodule of M, $t(X)$ be an X-cyclic submodule of X and let $\alpha : t(X) \to N$ be an R-monomorphism. Since $ts \in S$ and $ts(M) = t(X)$, $t(X)$ is an M-cyclic submodule of M. Since N is PM-P-injective, there exists an R-homomorphism $\hat{\alpha} : M \to N$ such that $\alpha = \hat{\alpha}\iota_2\iota_1$ where $\iota_2 : s(M) \to M$ and $\iota_1 : t(X) \to s(M)$ are the inclusion maps. Then $\hat{\alpha}\iota_2$ is the extension of α. This shows that N is PX-P-injective.

($\Leftarrow$) It is clear because M is an M-cyclic submodule of M. □

3.2.5 Lemma. *Let M be pseudo quasi-principally injective. If A is a direct summand of M, then A is PM-P-injective.*

Proof. Let A be a direct summand of M. Let $s \in S$ and $\alpha : s(M) \to A$ be an R-monomorphism. Let $\varphi : A \to M$ be the injection map. To show that $Ker(\varphi\alpha) = 0$. Let $s(m) \in Ker(\varphi\alpha)$. Then $\varphi\alpha(s(m)) = 0$. Since $\varphi(\alpha(s(m))) = \alpha(s(m)) + 0$, $\alpha(s(m)) = 0$. Hence $s(m) = 0$ because α is monic. Then $\varphi\alpha : s(M) \to M$ is an R-monomorphism. Since M is PQP-injective and $s(M)$ is an M-cyclic submodule of M, there exists an R-homomorphism $\hat{\alpha} : M \to M$ such that $\varphi\alpha = \hat{\alpha}\iota$ where $\iota : s(M) \to M$ is the inclusion map. Let $\pi : M \to A$ be the projection map. Then $\pi\varphi\alpha = \pi\hat{\alpha}i$.

Since $\pi\varphi = 1_A$, $\alpha = \pi\hat{\alpha}i$. Therefore $\pi\hat{\alpha}$ is an extension of α. This shows that *A is* pseudo *M*-principally injective. □

Let *M* be a right *R*-module and $S = End_R(M)$. Following [11], we write

$$W(S) = \left\{ w \in S : Ker(w) \subset^e M \right\}.$$

It is known that $W(S)$ is an ideal of *S*.

3.2.6 Proposition. *Let M be a PQ-P-injective module and* $S = End_R(M)$.

(1) *If* $S/W(S)$ *is regular, then* $J(S) = W(S)$.

(2) *If* $S/J(S)$ *is regular, then* $S/W(S)$ *is regular if and only if* $J(S) = W(S)$.

(3) *If* $Im(s) \subset^e M$ *where* $s \in S$*, then any R-monomorphism* $\varphi : s(M) \to M$ *can be extended to an R-monomorphism in S.*

Proof. (1) ($\supset$) Let $s \in W(S)$ and let $t \in S$. To show that $Ker(s) \cap Ker(1-ts) = 0$. Let $x \in Ker(s) \cap Ker(1-ts)$. Then $x \in Ker(s)$ and $x \in Ker(1-ts)$ so $s(x) = 0$ and $(1-ts)(x) = 0$. Hence $1(x) = t(s(x))$ so $x = 1(x) = t(s(x)) = 0$. Since $Ker(s) \subset^e M$, $Ker(1-ts) = 0$. Thus $S = S(1-ts)$ by Theorem 3.2.2. Since $1 \in S$, $1 \in S(1-ts)$. Write $1 = g(1-ts)$ for some $g \in S$. Then by Proposition 2.11.3, $s \in J(S)$. This shows that $W(S) \subset J(S)$. ($\subset$) Let $s \in J(S)$. Since $S/W(S)$ is regular, $s = s\alpha s$ for some $\alpha \in S/W(S)$ by Definition 2.12.1. Then $s - s\alpha s = 0 \in W(S)$. Hence $(1 - s\alpha)s = s - s\alpha s \in W(S)$, so $(1 - s\alpha)s \in W(S)$. By Proposition 2.11.3, we have $1 - s\alpha$ has an inverse. Let g be an inverse of $1 - s\alpha$. Thus $g(1 - s\alpha) = 1$. Then $s = 1s = g(1 - s\alpha)s \in W(S)$, so $s \in W(S)$. This shows that $J(S) \subset W(S)$.

(2) ($\Rightarrow$) By (1).

($\Leftarrow$) Since $S/J(S)$ is regular and $J(S) = W(S)$, $S/W(S)$ is regular.

(3) Let $\varphi : s(M) \to M$ be an *R*-monomorphism. Since *M* is *PQP*-injective module, there exists *R*-homomorphism $g : M \to M$ such that $\varphi = g\iota$ where $\iota : s(M) \to M$ is the inclusion map. Then $\varphi s = g\iota s = gs$. Let $x \in \mathrm{Im}(s) \cap Ker(g)$. Then $x \in \mathrm{Im}(s)$ and $x \in Ker(g)$. Hence $x = s(m)$ for some $m \in M$ and $g(x) = 0$. Thus $\varphi(s(m)) = g(s(m)) = g(x) = 0$, so $\varphi(s(m)) = 0$. Since φ is monic, $s(m) = 0$. Then $x = s(m) = 0$. This shows that $\mathrm{Im}(s) \cap Ker(g) = 0$. Since $\mathrm{Im}(s) \subset^e M$, $Ker(g) = 0$. Therefore g is an R-monomorphism. □

3.2.7. Lemma. *Let M be a pseudo quasi-principally injective module and $S = End_R(M)$.*

(1) *If $s(M)$ is a simple right R-module, $s \in S$, then Ss is a simple left S-module.*

(2) *If S is local, then $J(S) = \{ s \in S : Ker(s) \neq 0 \}$.*

Proof. **(1)** Let A be a nonzero submodule of Ss and $0 \neq \alpha s \in A$. Then $S\alpha s \subset A$. Suppose $Ker(\alpha) \cap s(M) \neq 0$. Since $s(M)$ is simple and $Ker(\alpha) \cap s(M) \subset s(M)$, $Ker(\alpha) \cap s(M) = s(M)$. Hence $s(M) \subset Ker(\alpha)$, so $\alpha s(M) = 0$. Thus $\alpha s = 0$, a contradiction so $Ker(\alpha) \cap s(M) = 0$. Then $Ker(\alpha s) = Ker(s)$. Hence $S\alpha s = Ss$ by Theorem 3.2.2. Since $S\alpha s \subset A \subset Ss$, $A = Ss$.

(2) Since S is local, $Ss \neq S$ for any $s \in J(S)$ by Proposition 2.11.6. To show that $J(S) = \{ s \in S : Ker(s) \neq 0 \}$. ($\subset$) Let $s \in J(S)$. To show that $Ker(s) \neq 0$. Suppose that $Ker(s) = 0$. Define $\alpha : s(M) \rightarrow M$ given by $\alpha(s(m)) = m$ for any $m \in M$. Let $0 = s(m) \in s(M)$. Then $m \in Ker(s) = 0$, so $m = 0$. Hence $\alpha(s(m)) = m = 0$. This shows that α is well-defined. Let $s(m_1)$, $s(m_2) \in s(M)$ and $r \in R$. Then $\alpha(s(m_1)r + s(m_2)) = \alpha(s(m_1 r) + s(m_2)) = \alpha(s(m_1 r + m_2)) = m_1 r + m_2$ $= \alpha(s(m_1))r + \alpha(s(m_2))$. This shows that α is an R-homomorphism. To show that α is an R-monomorphism. That is $Ker(\alpha) = 0$. Let $s(m) \in Ker(\alpha)$. Then $\alpha(s(m)) = 0$, so $m = \alpha(s(m)) = 0$. Hence $s(m) = s(0) = 0$. Since M is pseudo quasi-principally injective, there exists an R-homomorphism $\beta : M \rightarrow M$ such that $\alpha = \beta\iota$ where $\iota : s(M) \rightarrow M$ is the inclusion map. It follows that $\beta s = \beta\iota s = \alpha s = 1_M$ and hence $\beta s = 1_M$, so $Ker(\beta s) = Ker(1_M)$. Then $S = S\beta s$ by Theorem 3.2.2. Since $S\beta s \subset Ss$, $S = Ss$ which is a contradiction. This shows that $J(S) \subset \{ s \in S : Ker(s) \neq 0 \}$. ($\supset$) Let $s \in \{ s \in S : Ker(s) \neq 0\}$. Since S is local, $J(S) = \{ s \in S : Ss \neq S \}$. To show that $Ss \neq S$. Suppose that $Ss = S$. Then $fs = 1_M$ for some $f \in S$. Since $Ker(1_M) = 0$, $Ker(fs) = 0$. We have $Ker(s) \subset Ker(fs)$. Then $Ker(s) = 0$, a contradiction. □

An R-module M is called π-injective [14] if for all submodule U and V of M with $U \cap V = 0$, there exists $f \in S$ with U $\subset$ $Ker(f)$ and V $\subset$ $Ker(1{-}f)$. A nonzero module M is called *uniform* if every non-zero submodule of M is essential in M.

3.2.8 Proposition. Let M be a PQ-P-injective module.

(1) If S is local and M is π-injective, then M is uniform.

(2) If M is uniform, then $Z(S_S) \subset J(S)$.

Proof. (1) Let U and V be submodules of M such that $U \cap V = 0$. Since M is π-injective, there exists $f \in S$ with $U \subset Ker(f)$ and $V \subset Ker(1-f)$. Since S is local, we have $f \in J(S)$ or $1-f \in J(S)$, by Proposition 2.11.6. If $f \in J(S)$, then $1-f$ has an inverse by Proposition 2.11.3. Hence $1-f$ is monic, $Ker(1-f) = 0$. Since $V \subset Ker(1-f)$, $V = 0$. Otherwise $U = 0$.

(2) Let $s \in Z(S_S)$ and $0 \neq t \in S$. Since $r_S(s) \subset^e S$, there exists $f \in S$ such that $0 \neq ft \in r_S(s)$ by Proposition 2.2.4. Then $s(ft) = 0$. If $Ker(s) = 0$, then s is monic. Since $s(ft) = 0$, $ft = 0$, a contradiction. This shows that $Ker(s) \neq 0$. Since M is uniform, $Ker(s) \subset^e M$. Let $t \in S$. To show that $Ker(s) \cap Ker(1-ts) = 0$. Let $x \in Ker(s) \cap Ker(1-ts)$. Then $x \in Ker(s)$ and $x \in Ker(1-ts)$ so $s(x) = 0$ and $(1-ts)(x) = 0$. Hence $1(x) = t(s(x))$ so $x = 1(x) = t(s(x)) = t(0) = 0$. Since $Ker(s) \subset^e M$, $Ker(1-ts) = 0$. Then $Ker(1-ts) = Ker(1_M)$. Thus $S = S(1-ts)$ by Theorem 3.2.2. Since $1 \in S$, $1 \in S(1-ts)$. Write $1 = g(1-ts)$ for some $g \in S$. This shows that s $\in J(S)$. □

Following [9], a ring R is called semiregular if $R/J(R)$ is regular and idempotent can be lifted modulo $J(R)$, equivalently, R is semiregular if and only if each element $a \in R$, there exists $e^2 = e \in Ra$ such that $a(1 - e) \in J(R)$.

3.2.9. Theorem. *For a pseudo quasi-principally injective module M, if S is semiregular, then for every $s \in S \setminus J(S)$, there exists a nonzero idempotent $\alpha \in Ss$ such that $Ker(s) \subset Ker(\alpha)$ and $Ker(s(1-\alpha)) \neq 0$.*

Proof. Let $s \in S \setminus J(S)$. Since S is a semiregular ring, there exists $\alpha^2 = \alpha \in Ss$ such that $s(1-\alpha) \in J(S)$. Then $\alpha^2 = \alpha = fs$ for some $f \in S$. If $\alpha = 0$, then $s = s(1-0) = s(1-\alpha) \in J(S)$, a contradiction. This shows that $\alpha \neq 0$. Let $x \in Ker(s)$. Then $s(x) = 0$. Hence $\alpha(x) = fs(x) = f(s(x)) = f(0) = 0$, so $x \in Ker(\alpha)$. This shows that $Ker(s) \subset Ker(\alpha)$. Suppose that $Ker(s(1-\alpha)) = 0$. Then $Ker(s(1-\alpha)) = Ker(1_M)$. Since M is PQ-P-injective module, by Theorem 3.2.2, $Ss(1-\alpha) = S$. We have $1_M \in S$, so $gs(1-\alpha) = 1_M$ for some $g \in S$. Then $gs - gs\alpha = 1_M$. Hence $(gs - gs\alpha)\alpha = 1_M\alpha$, so $gs\alpha - gs\alpha^2 = \alpha$. Thus $gs\alpha - gs\alpha^2 = gs\alpha - gs\alpha = 0$. It follows that $\alpha = 0$, a contradiction. This shows that $Ker(s(1-\alpha)) \neq 0$. □

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Appendix

Conference Proceeding

Paper Title “On PPQ-injective and PQP-injective Modules”

The 5th Conference on Fixed Point Theory and Applications.

July 8 – 9, 2011.

At Lampang Rajabhat University,

Lampang, Thailand.

The 5th Annual Conference on

Fixed Point Theory an Applications

at Lampang Rajabhat University, Lampang, Thailand

July 8 - 9, 2011

Abstracts

In celebration of the 40th anniversary of Lampang Rajabhat University

Content

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THE 5th CONFERENCE ON FIXED POINT THEORY AND APPLICATIONS

Faculty of Science, Lampang Rajabhat University July 8-9, 2011

ON PPQ-INJECTIVE AND PQP-INJECTIVE MODULES

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Let M be a right $R-$module. The module M is called *pseudo principally quasi-injective* (briefly, PPQ*-injective*) if, it is pseudo principally M-injective [2]. The module M is called *pseudo quasi-principally injective* (briefly, PQP*-injective*) if, it is pseudo M-principally injective [1]. In this paper, we give some characterizations and properties of the two classes of modules.

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